Exercise 7 for May 2nd

Exercise. Let $(X_i)_{i\geq 1}$ be a sequence i.i.d. random variables with $\mathbb{E}[X_1] = 0$ and $\mathbb{P}(X_1 \geq u) \sim c/u^\beta$ as $u \to \infty$ with $\beta > 2$. Set $S_n = X_1 + \cdots + X_n$. Let $(x_n)_{n\geq 1}$ be a sequence such that $\liminf_{n\to\infty} x_n/n > 0$.

Obtain a limit theorem for $\max(X_1, \ldots, X_n)$ under $\mathbb{P}(\cdot|S_n \geq x_n)$ as $n \to \infty$.

Set $M_n = \max(X_1, \ldots, X_n)$. <u>Intuition</u>: By the one big-jump principle we should have $\mathbb{P}(M_n > u x_n) \simeq n \mathbb{P}(X_1 > u x_n)$ for n large. In addition $\mathbb{P}(S_n > u x_n) \sim n \mathbb{P}(X_1 > u x_n)$

Thus we expect, for $u > 1$, $\frac{\mathbb{P}(M_n > u x_n)}{\mathbb{P}(S_n > x_n)} \xrightarrow[n\to\infty]{} \frac{1}{u^\beta}$

We show that $\frac{M_n}{x_n}$ converges in distribution to J with density $\frac{\beta}{u^{\beta+1}} \mathbb{1}_{u \geq 1} \, du$

As in the lecture, write $M_n = S_n - \hat{X}_1 + \cdots + \hat{X}_{n-1}$

By Slutsky's lemma, it is enough to show that under $\mathbb{P}(\cdot \,|\, S_n > x_n)$:

(a) $\frac{S_n}{x_n} \xrightarrow{(d)} J$

(b) $\frac{\hat{X}_1 + \cdots + \hat{X}_{n-1}}{x_n} \xrightarrow{\mathbb{P}} 0$

For (a), this follows by writing $\frac{\mathbb{P}(S_n > u x_n)}{\mathbb{P}(S_n > x_n)} \sim \frac{n \mathbb{P}(X_1 > u x_n)}{n \mathbb{P}(X_1 > x_n)} \xrightarrow[n\to\infty]{} \frac{1}{u^\beta}$ for $u > 1$.

For (b), this follows from the fact that under $\mathbb{P}(\cdot \,|\, S_n > x_n)$, $\hat{X}_1, \ldots, \hat{X}_{n-1}$ are asymptotically iid with law X_1, and $\underbrace{\frac{X_1 + \cdots + X_{n-1}}{n}}_{\xrightarrow{\mathbb{P}} 0} \cdot \underbrace{\frac{n}{x_n}}_{\limsup < \infty} \xrightarrow{\mathbb{P}} 0$.