Exercise 8 for May 16th

Exercise. Fix $p \in [1/2, 1]$. Set $\mu(k) = p(1-p)^k$ for $k \geq 0$. Let $\mathcal{T}_n$ be a B_μ random tree conditioned on having n vertices. Show that $\mathcal{T}_n$ follows the uniform distribution on the set of all plane trees with n vertices.

It is enough to show that if T_1, T_2 are plane trees with n vertices, then $\mathbb{P}(\mathcal{T} = T_1) \overset{(*)}{=} \mathbb{P}(\mathcal{T} = T_2)$ with $\mathcal{T}$ a B_μ-tree. Indeed, we then have $\mathbb{P}(\mathcal{T}_n = T_1) = \frac{\mathbb{P}(\mathcal{T} = T_1)}{\mathbb{P}(|\mathcal{T}| = n)} = \frac{\mathbb{P}(\mathcal{T} = T_2)}{\mathbb{P}(|\mathcal{T}| = n)} = \mathbb{P}(\mathcal{T}_n = T_2)$,

which shows that $\mathcal{T}_n$ is uniform on the set of plane trees with n vertices

To show $(*)$ the key observation is that if T has n vertices, $\sum_{u \in T} k_u(T) = n-1$. Then:

$\mathbb{P}(\mathcal{T} = T_1) = \prod_{u \in T_1} p(1-p)^{k_u(T_1)} = p^n (1-p)^{n-1}$ which only depends on n.

Then $\mathbb{P}(\mathcal{T} = T_1) = \mathbb{P}(\mathcal{T} = T_2)$.