Exercise 9 for May 23th

Exercise. Let μ be a subcritical offspring distribution (that is $\sum_{k=0}^{\infty} k\mu(k) < 1$) such that $G_\mu(x) = \sum_{k=0}^{\infty} \mu(k)x^k$ has ar infinite radius of convergence. Show that there exists $b > 0$ such that $\hat{\mu}$ defined by

$$\hat{\mu}(k) = \frac{b^k \mu(k)}{G_\mu(b)}$$

is critical (that is $\sum_{k=0}^{\infty} k\mu(k) = 1$).

This amounts to showing that $\exists b>0$ s.t $\frac{b\, G_\mu'(b)}{G_\mu(b)} = 1 \iff b\, G_\mu'(b) = G_\mu(b)$

Set $F(x) = x\, G_\mu'(x) - G_\mu(x)$

$$= \sum_{k=0}^{\infty} k\mu(k)\, x^k - \sum_{k=0}^{\infty} \mu(k) x^k$$

Observe that $F(0) = -\mu(0) < 0$, that F is continuous and that $\lim_{+\infty} F = +\infty$

Indeed, let $k_0 > 1$ be an integer with $\mu(k_0) > 0$ (which exists since μ is subcritical and is a probability measure). Then

$$F(x) = \underbrace{\mu(k_0)\, x^{k_0} - \mu(0)}_{x \to \infty} + \underbrace{\sum_{\substack{k=1 \\ k \neq k_0}}^{\infty} (k-1)\mu(k)\, x^k}_{\geq 0} + \underbrace{(k_0 - 2)\,\mu(k_0)\, x^{k_0}}_{\geq 0}$$

The result follows.