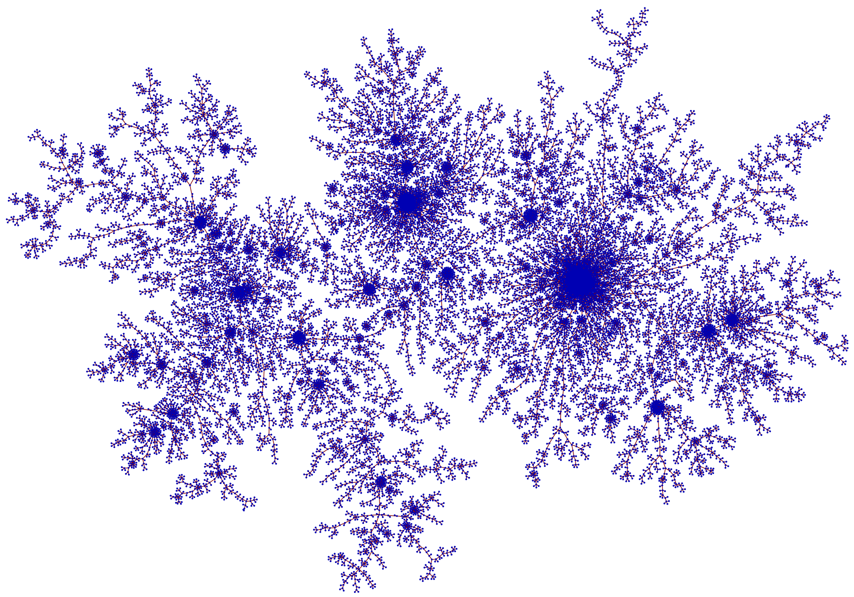
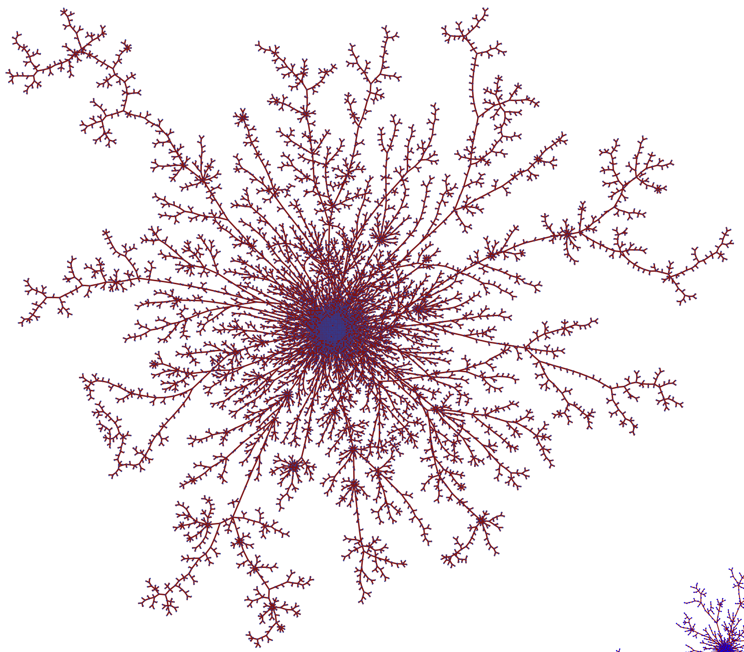
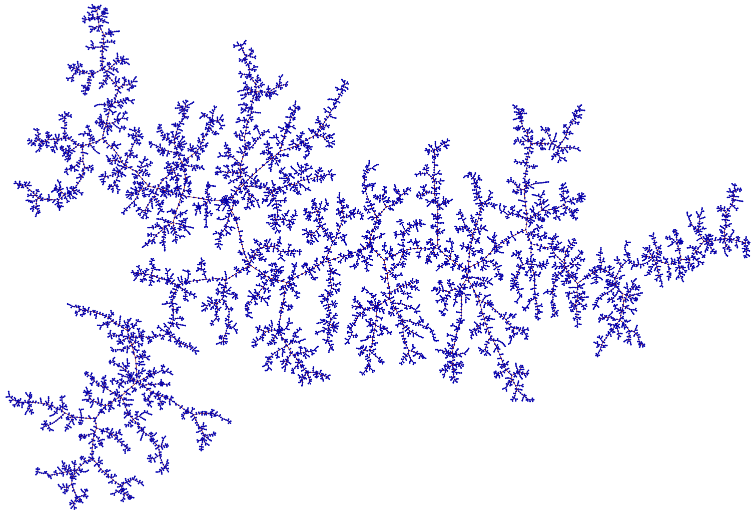
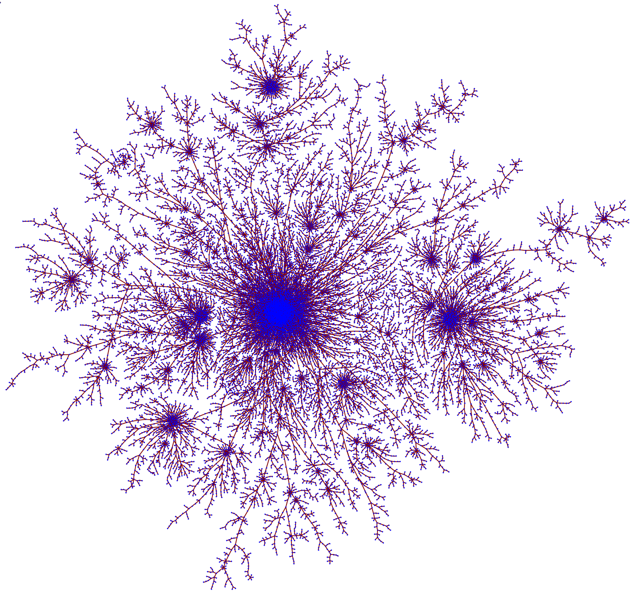
Limits of large random trees









Igor Kortchemski

ETH Zürich

Condensation Phenomena in Random Trees – Spring 2024

Context

Understand the geometry and the structure of large random trees by studying their scaling limits.

Motivation for studying limits

Let $(X_n)_{n \geqslant 1}$ be "discrete" objects converging towards a "limiting" object X:

$$X_n \quad \underset{n \to \infty}{\longrightarrow} \quad X.$$

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- *From the continuous world to the discrete world:* if a property $\mathcal{P}$ is satisfied by X and passes to the limit, X_n satisfies "approximately" $\mathcal{P}$ for n large.

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- *From the continuous world to the discrete world:* if a property $\mathcal{P}$ is satisfied by X and passes to the limit, X_n satisfies "approximately" $\mathcal{P}$ for n large.
- *Universality:* if $(Y_n)_{n \geqslant 1}$ is another sequence of objects converging towards X, then X_n and Y_n share approximately the same properties for n large.

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$$X_n \quad \underset{n\to\infty}{\overset{(d)}{\longrightarrow}} \quad X \qquad \text{implies} \qquad G(X_n) \quad \underset{n\to\infty}{\overset{(d)}{\longrightarrow}} \quad G(X)$$

for every continuous function $G : Z \to \mathbb{R}$.

Outline

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Stack triangulations (Albenque, Marckert)

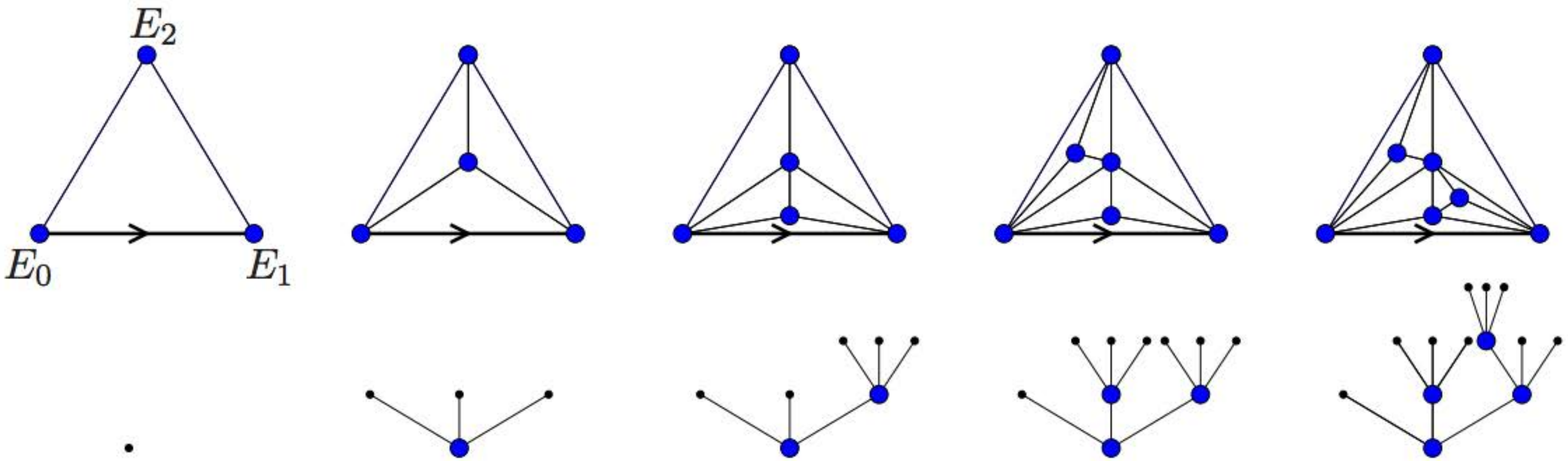


Figure 8: Construction of the ternary tree associated with an history of a stack-triangulation

Dissections (Curien, K.)

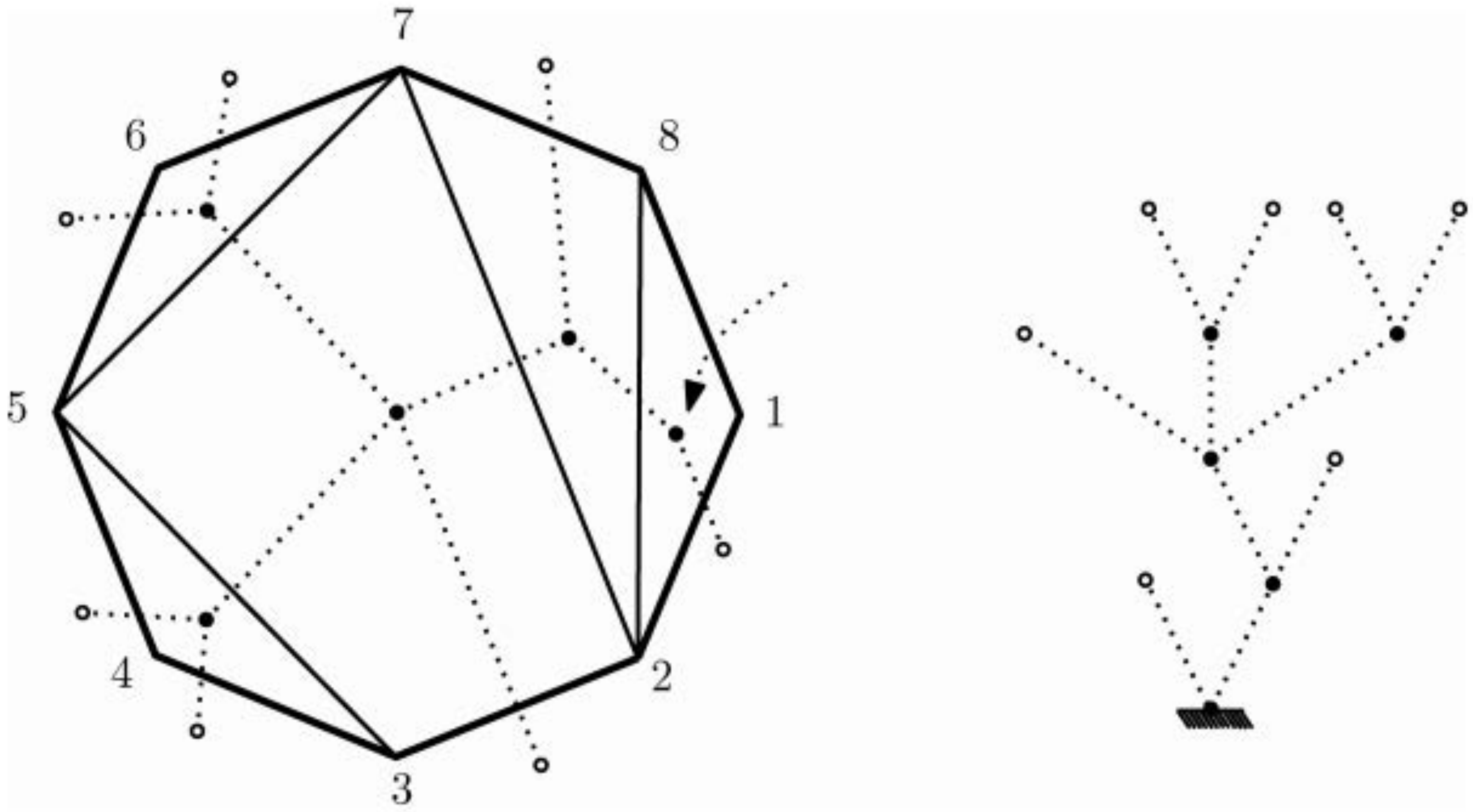


Fig. 4. The dual tree of a dissection of P_8, note that the tree has 7 leaves.

Maps (Schaeffer)

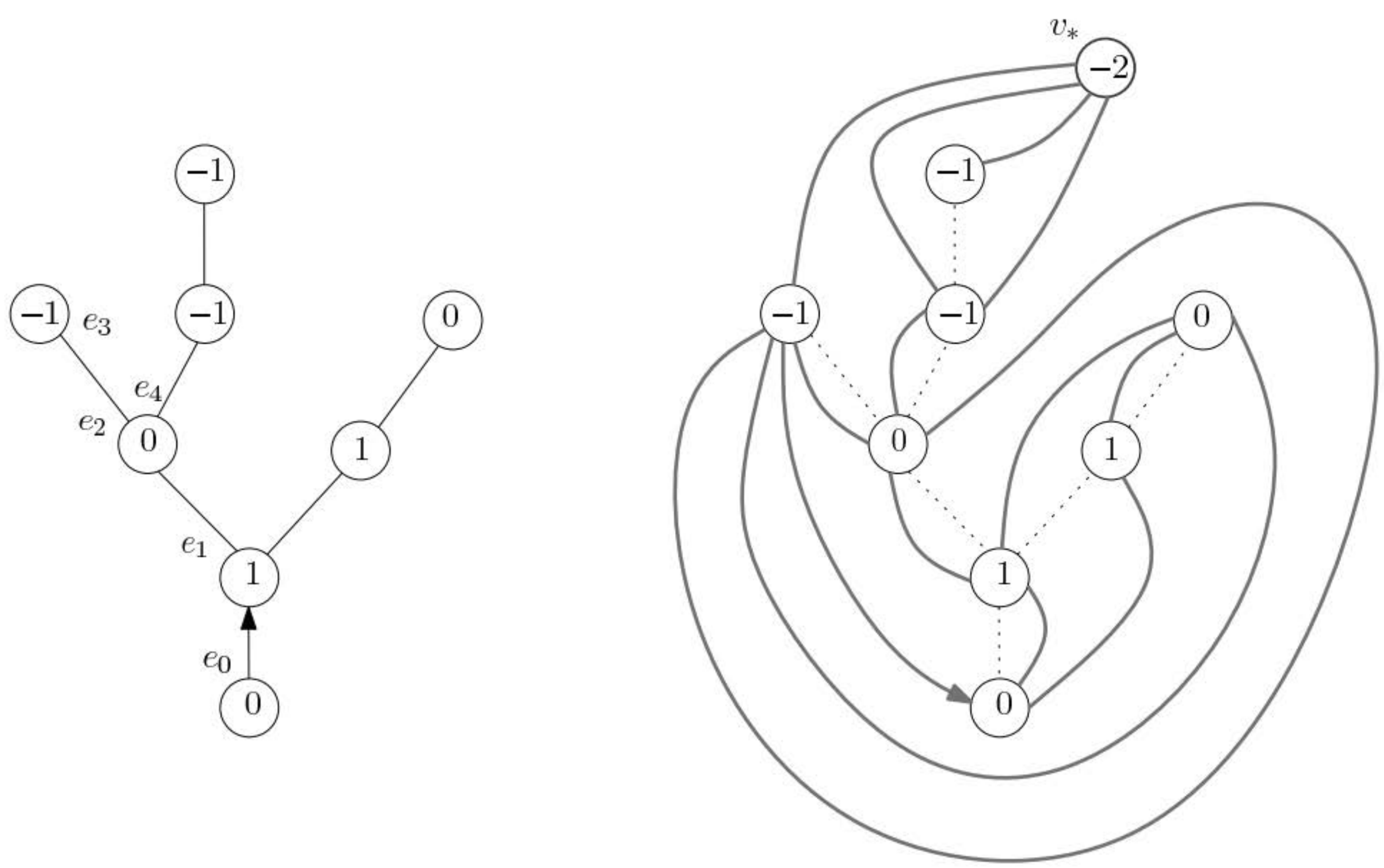
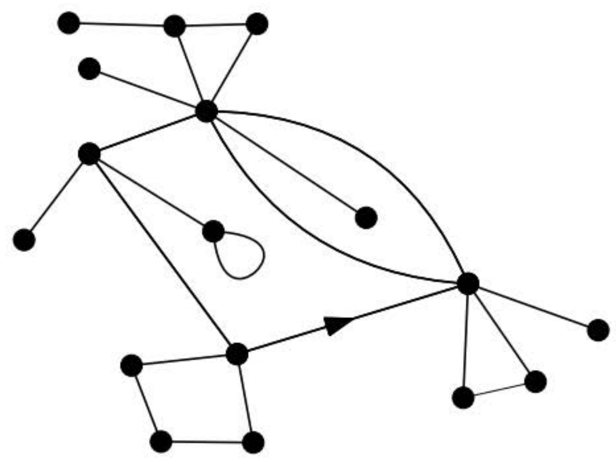
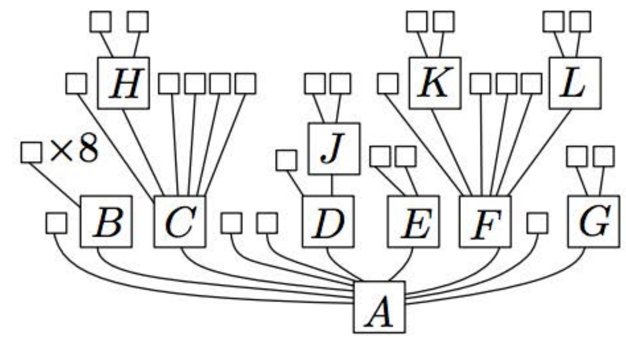


FIGURE 6. Illustration of the Cori-Vauquelin-Schaeffer bijection, in the case $\epsilon = 1$. For instance, e_3 is the successor of e_0, e_2 the successor of e_1, and so on.

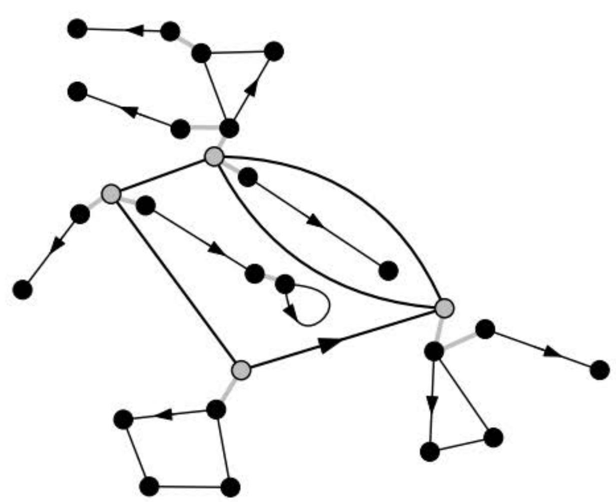
Maps (Addario-Berry)



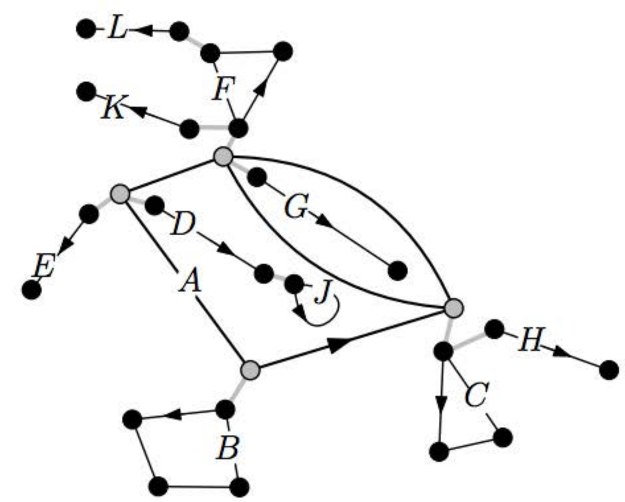
(A) A map M.



(B) The tree T_M. Tiny squares represent trivial blocks.

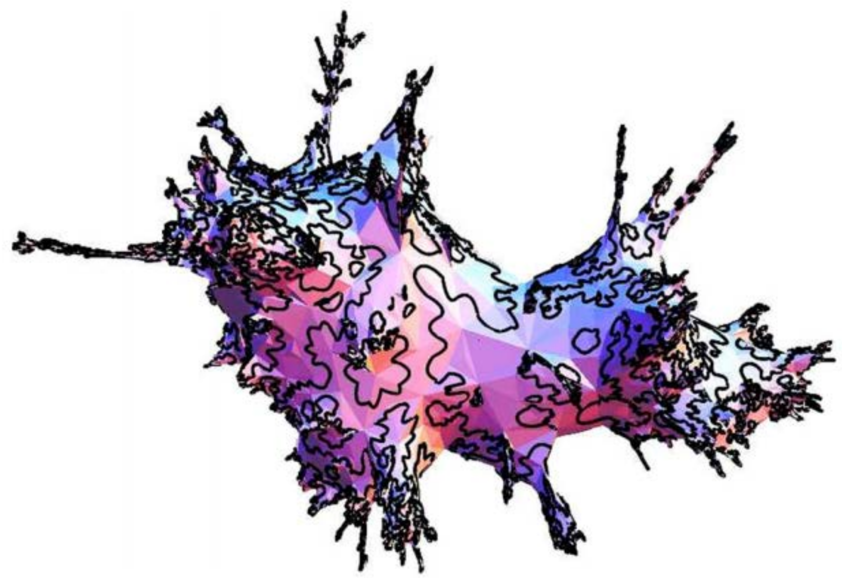


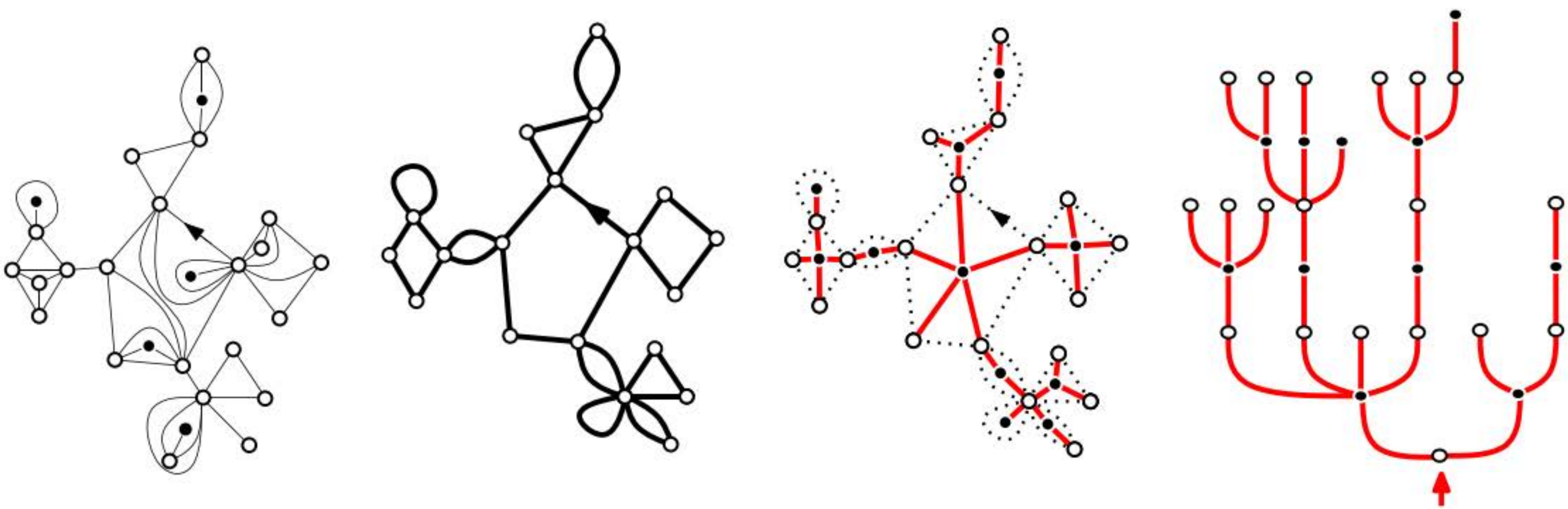
(C) The decomposition of M into blocks. Blocks are joined by grey lines according to the tree structure. Root edges of blocks are shown with arrows.



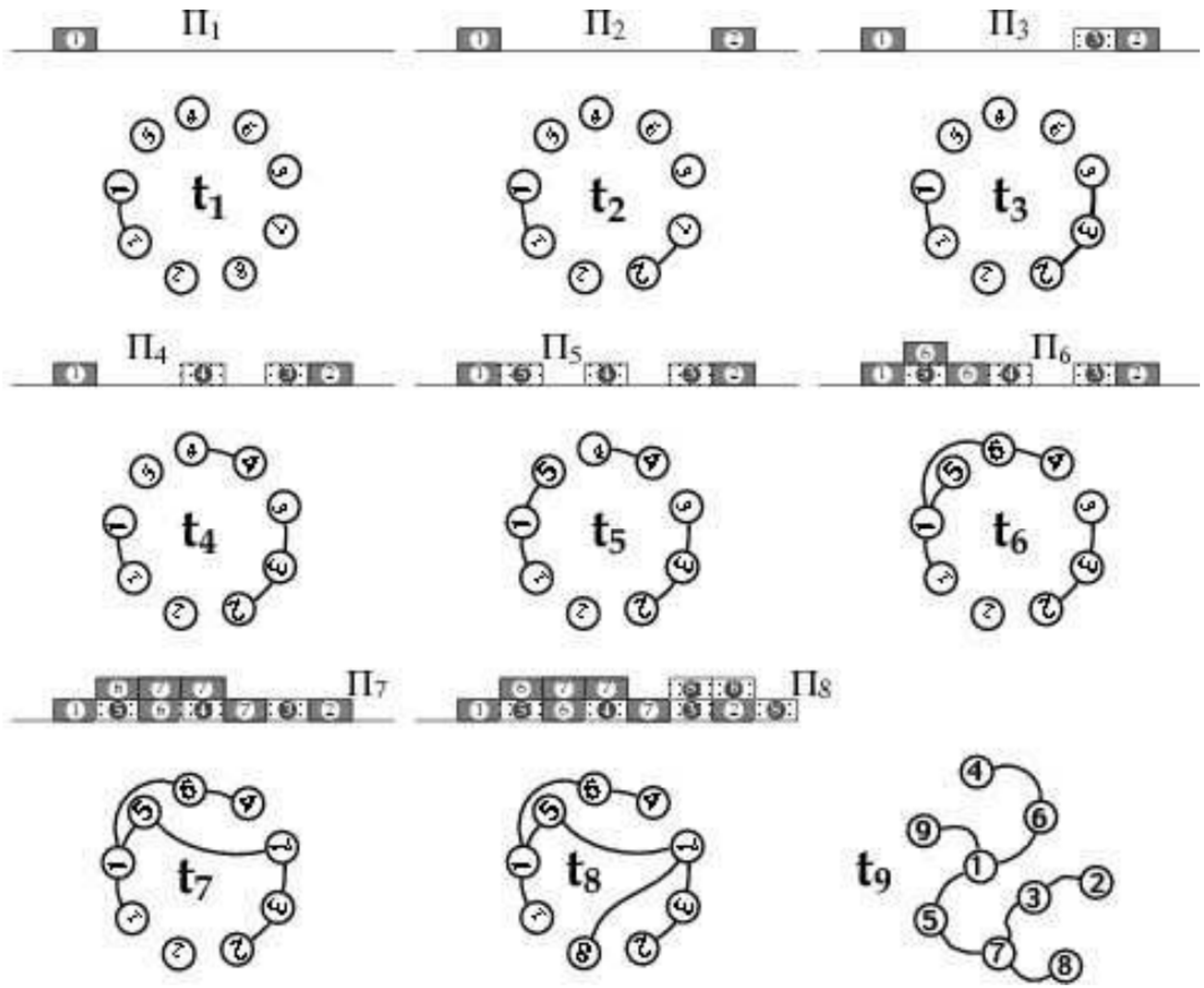
(D) The correspondence between blocks and nodes of T_M. Non-trivial blocks receive the alphabetical label (from A through L) of the corresponding node.

Maps with percolation (Curien, K.)





Parking functions (Chassaing, Louchard)



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Recall that in a Bienaymé tree, every individual has a random number of children (independently of each other) distributed according to μ (offspring distribution).

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What does a large size-conditioned Bienaymé tree look like, near the root?

Local limits

We will consider two regimes:

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💡 These regimes actually cover all the cases. Indeed, if $c < \rho_\mu$, two BGW trees with offspring distributions μ and μ_c, defined by

$$\mu_c(k) = \frac{1}{G_\mu(c)} c^k \mu(k), \qquad k \geqslant 0,$$

when conditioned on having n vertices, have the same distribution (Kennedy '75).

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Theorem (Kesten '87, Janson '12, Abraham & Delmas '14)

The convergence

$$\mathcal{T}_n \quad \xrightarrow[n\to\infty]{(d)} \quad \mathcal{T}_\infty$$

holds in distribution for the local topology, where $\mathcal{T}_\infty$ is the infinite Bienaymé tree conditioned to survive.

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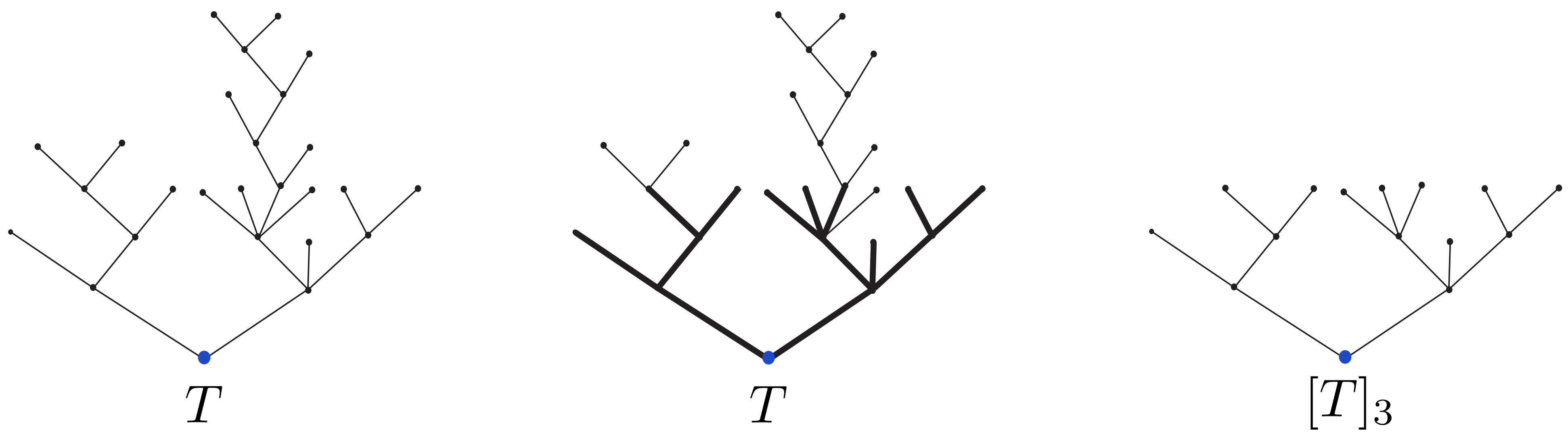
⤳ This means that $[\mathcal{T}_n]_k \to [\mathcal{T}_\infty]_k$ in distribution, where $[T]_k$ denotes the subtree of T obtained by keeping the first k children on the first k generations:



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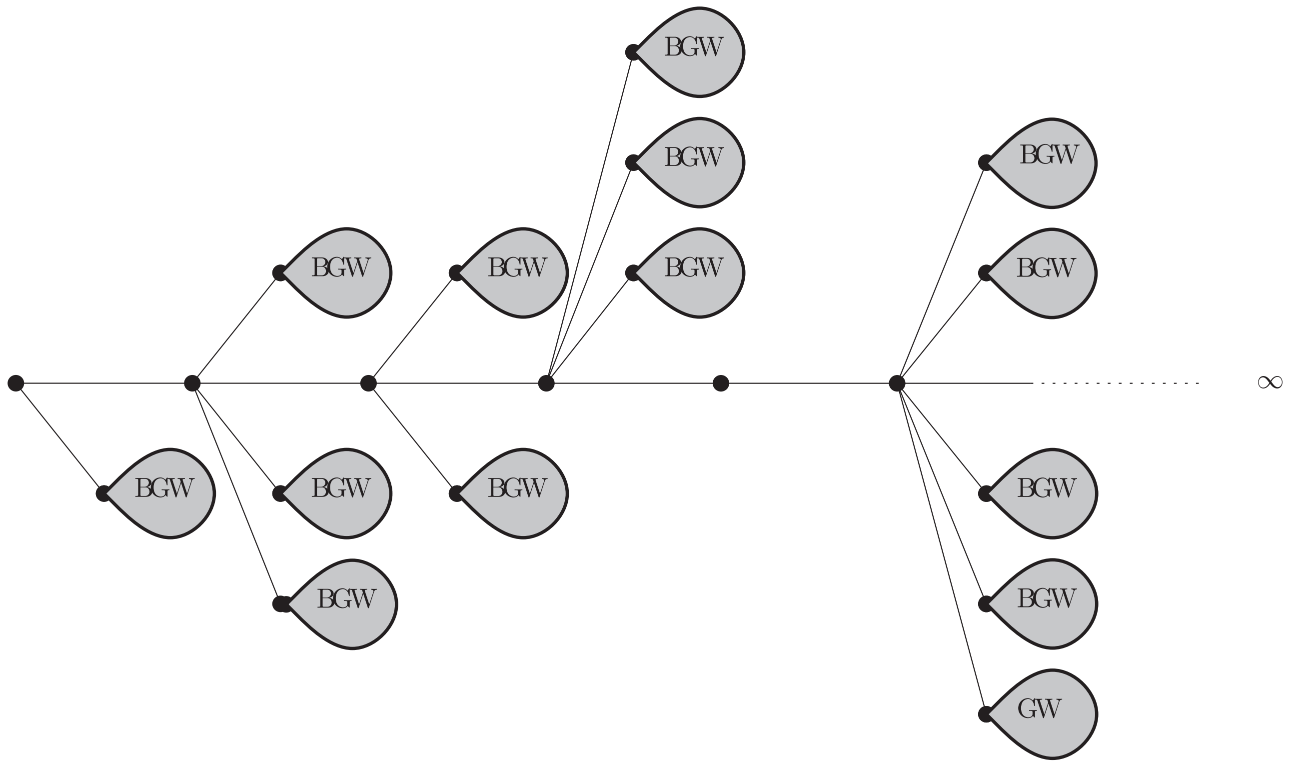

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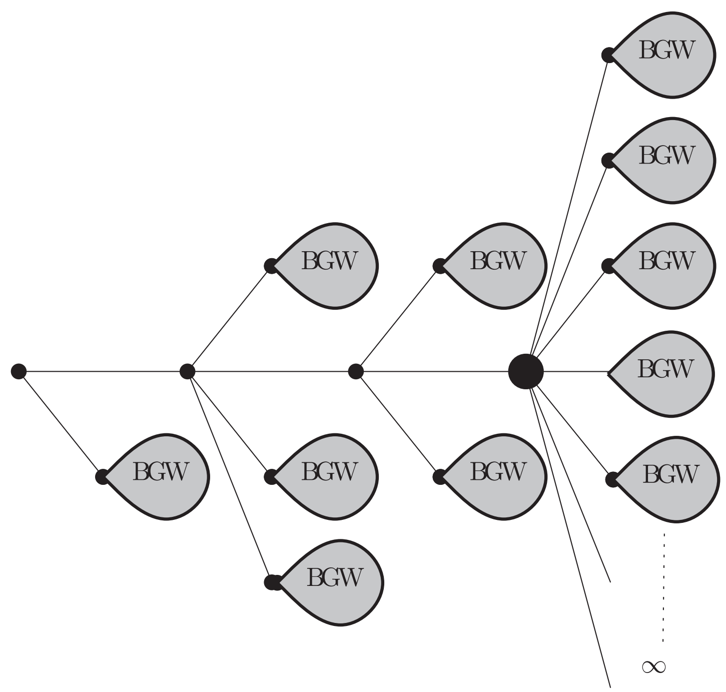

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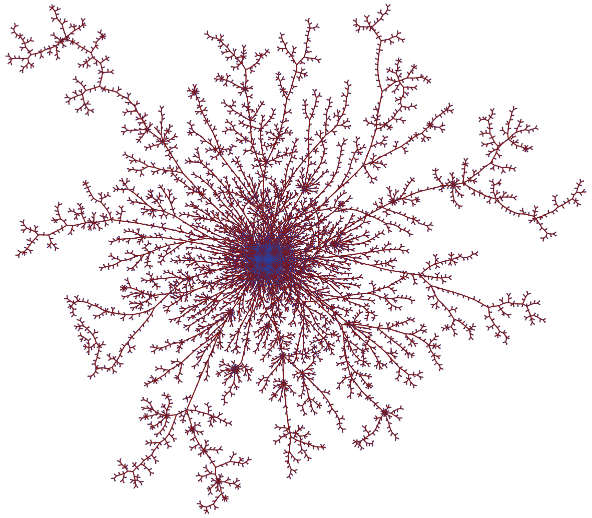
What does a large Bienaymé tree look like, globally?

I have simulated and drawn a uniform plane tree with 10000 vertices. What did I get?



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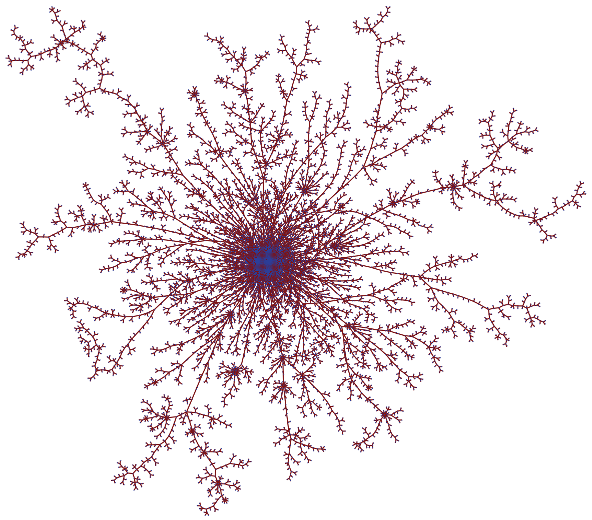

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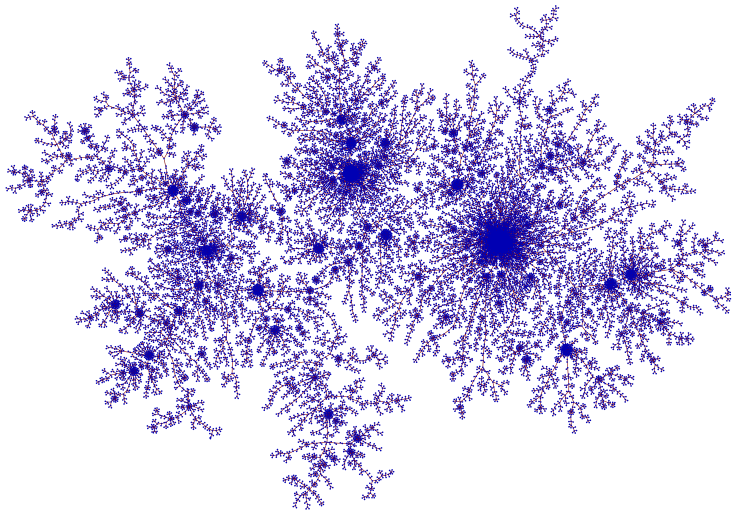

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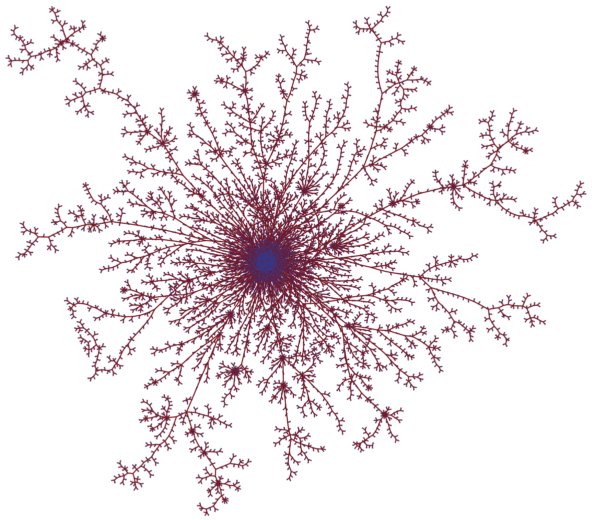

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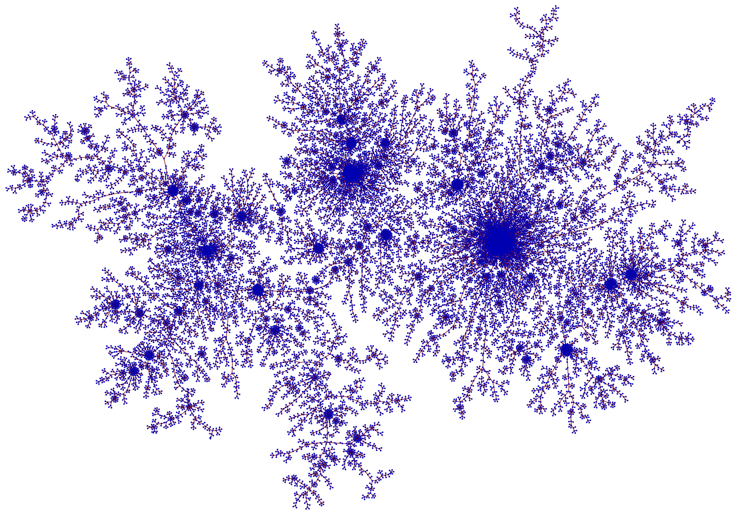

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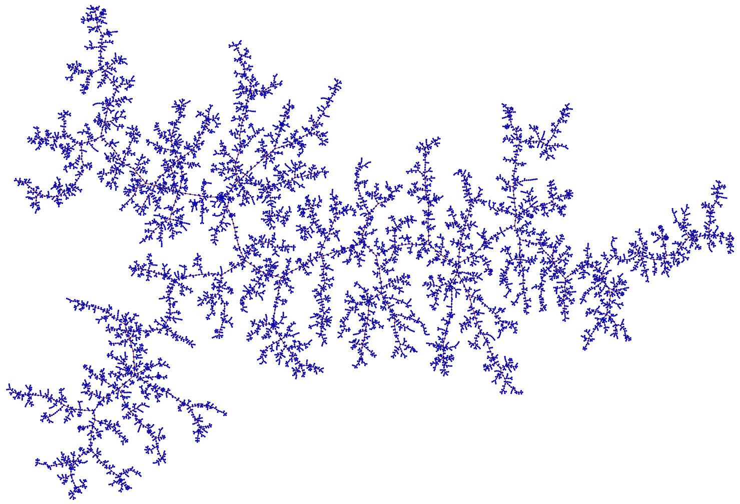

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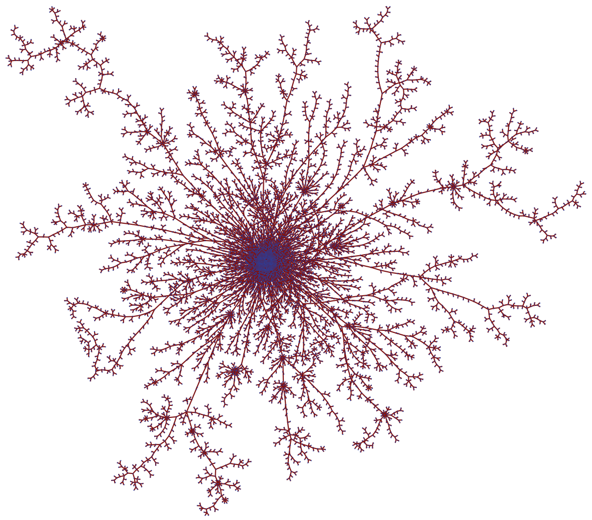

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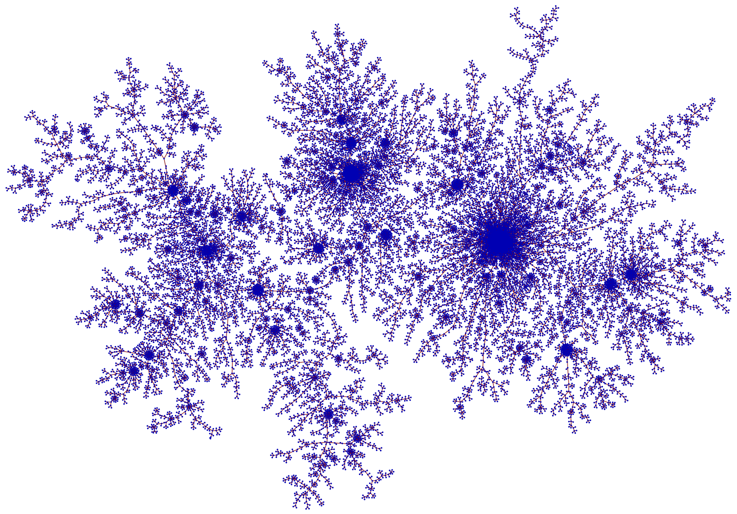

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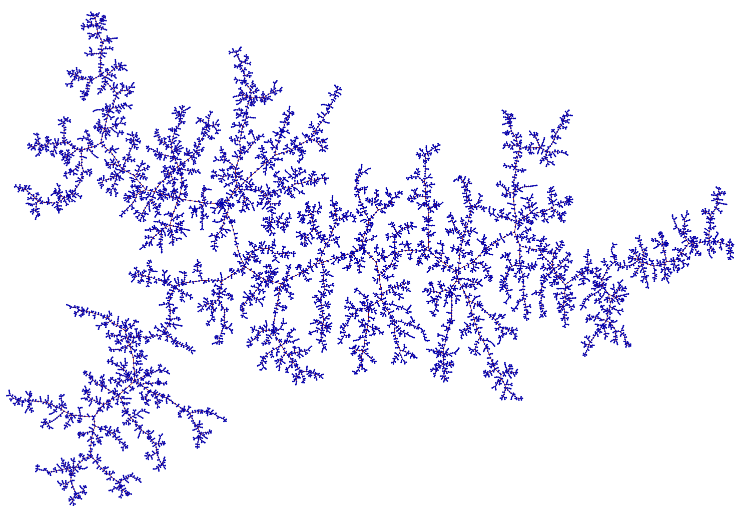

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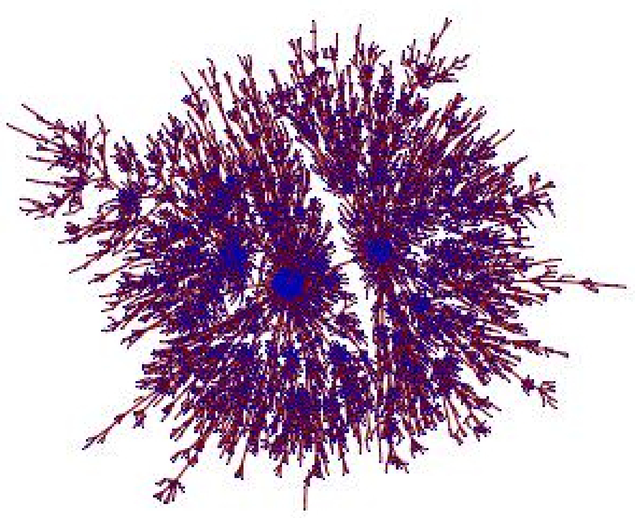

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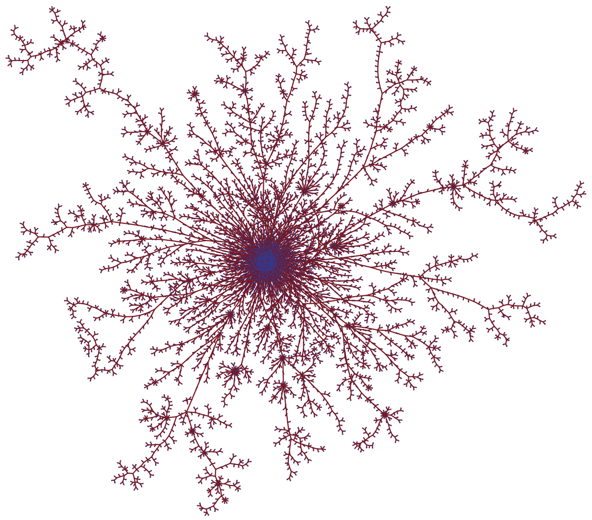

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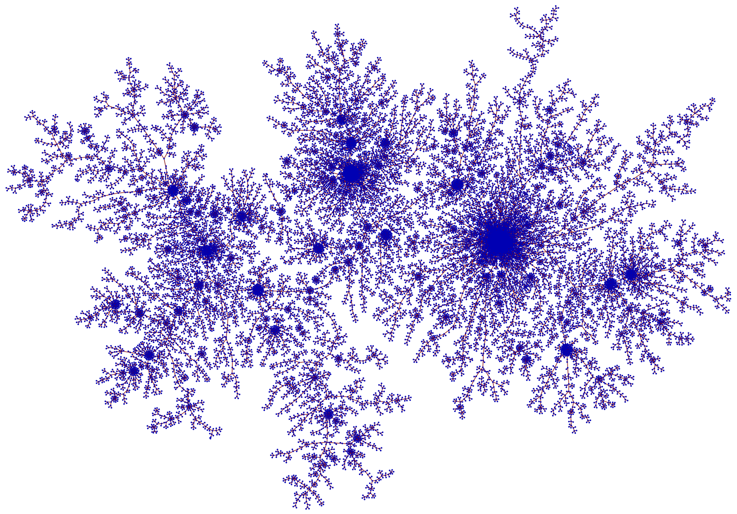

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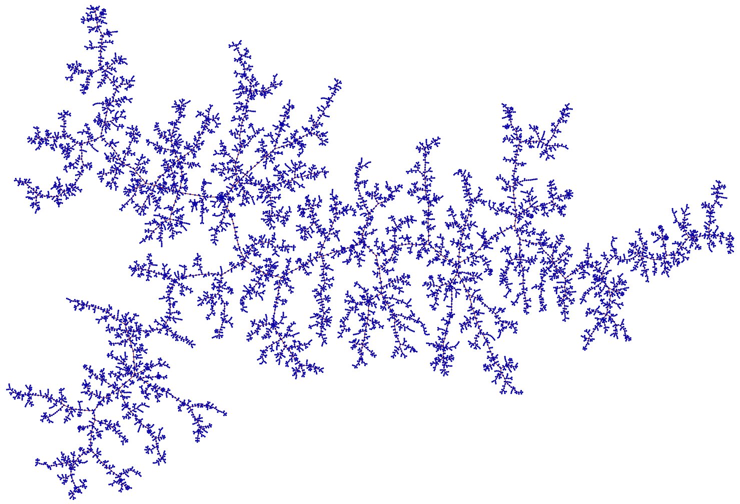

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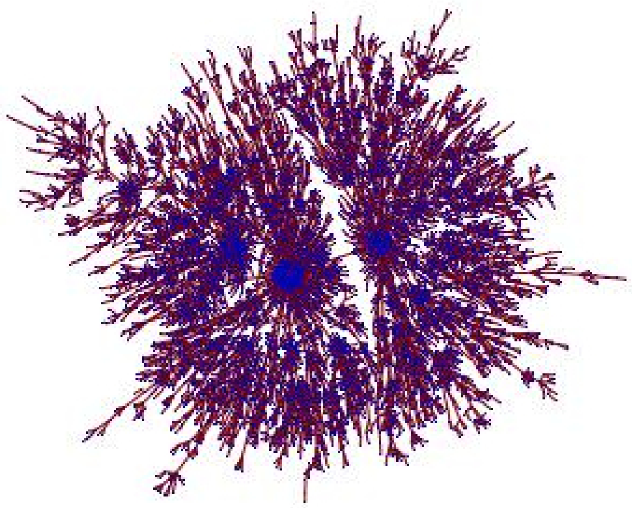

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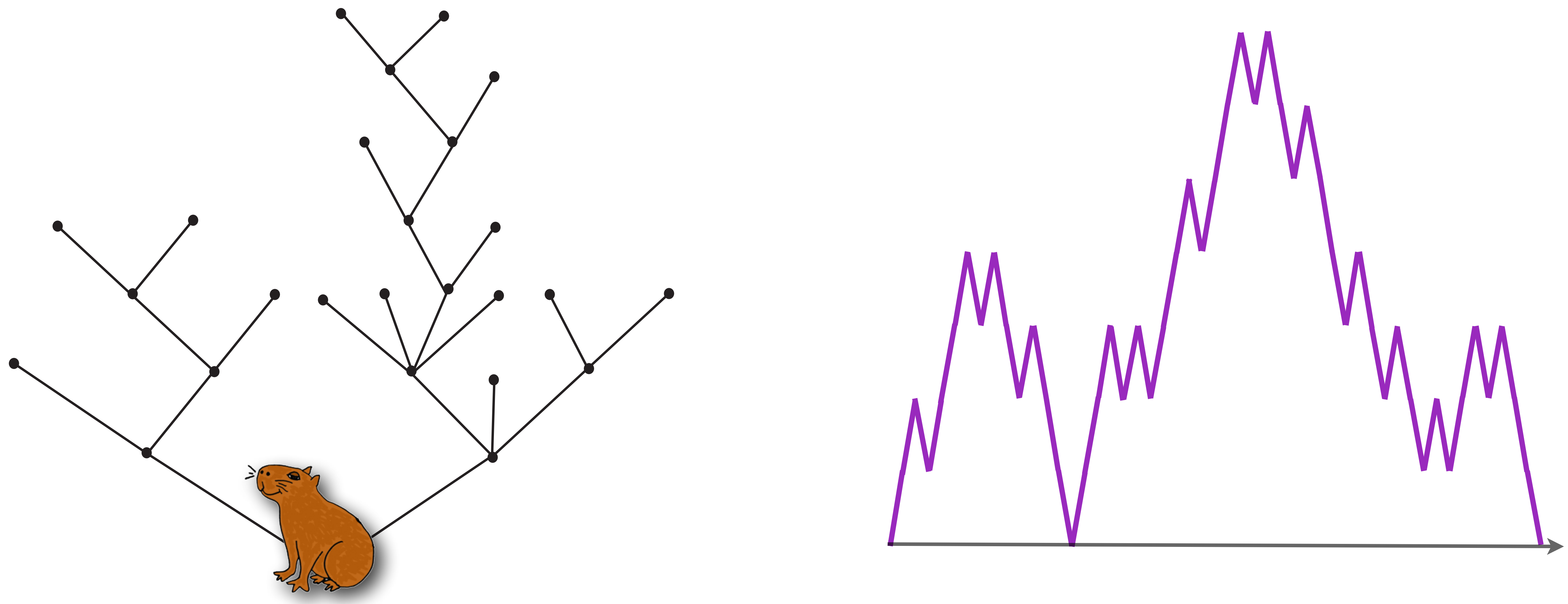
We shall code plane trees by functions.

Coding trees by functions

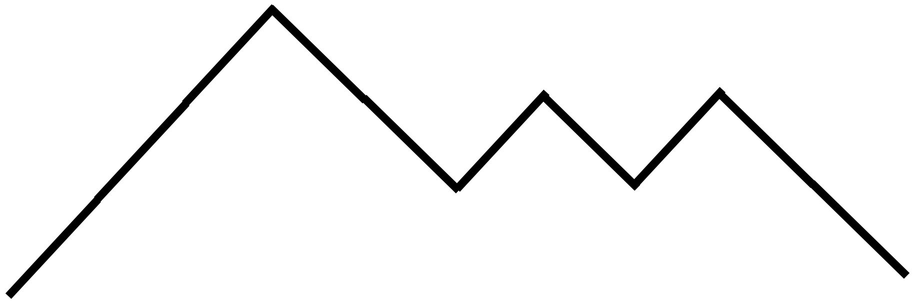
Contour function of a tree

Define the contour function of a tree:



Coding trees by contour functions

Knowing the contour function, it is easy to recover the tree.



Scaling limits

Scaling limits: finite variance

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Theorem (Aldous '93)

Let σ^2 be the variance of μ. Let $t \mapsto C_t(\mathcal{T}_n)$ be the contour function of $\mathcal{T}_n$. Then:

$$\left(\frac{1}{\sqrt{n}} C_{2nt}(\mathcal{T}_n) \right)_{0 \leqslant t \leqslant 1} \quad \xrightarrow[n \to \infty]{(d)}$$

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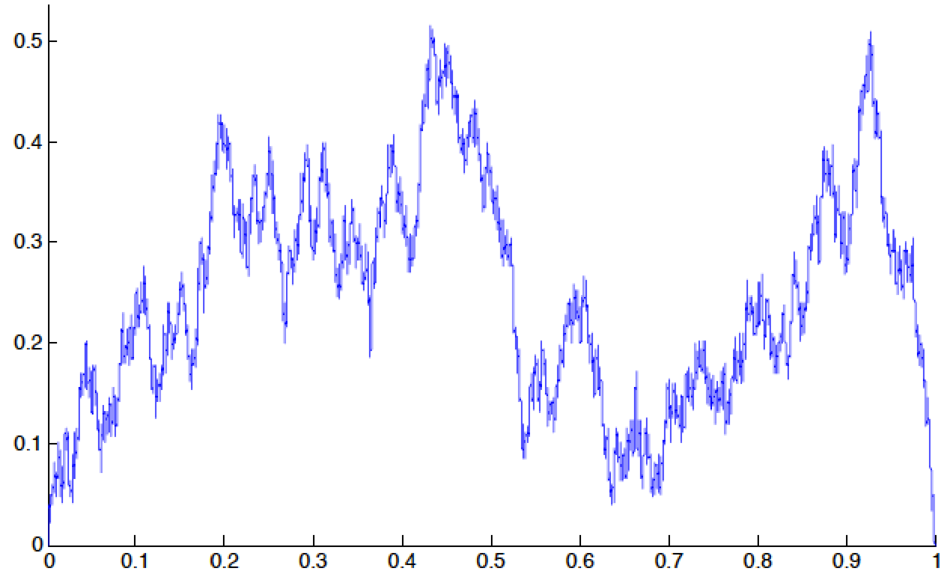

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Do the discrete trees converge to a continuous tree?

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Yes, if we view trees as compact metric spaces by equiping the vertices with the graph distance!

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Let X, Y be two subsets of the **same** metric space Z.

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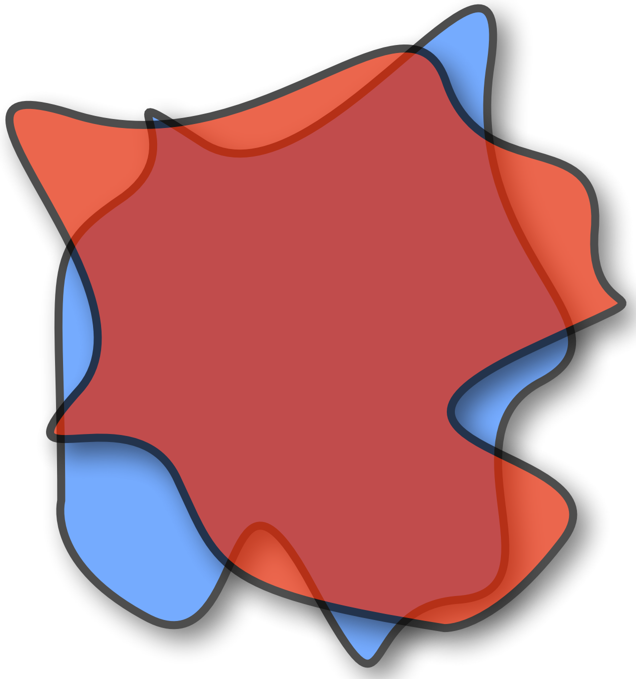

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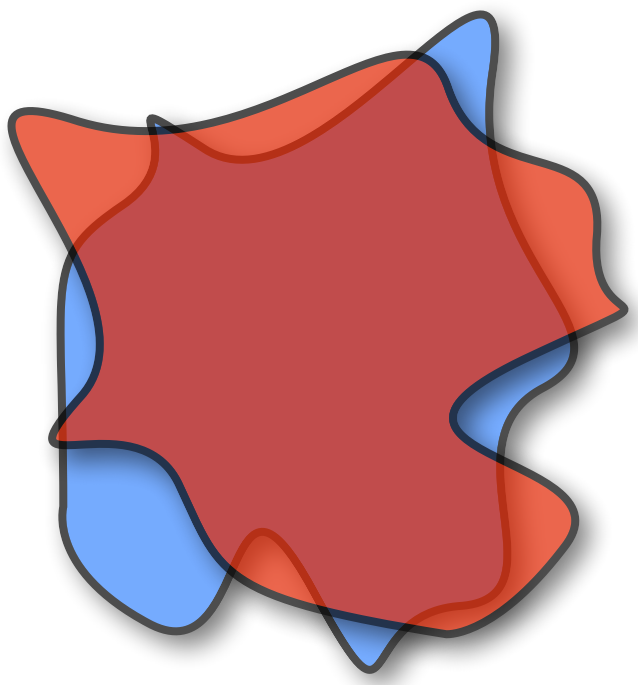

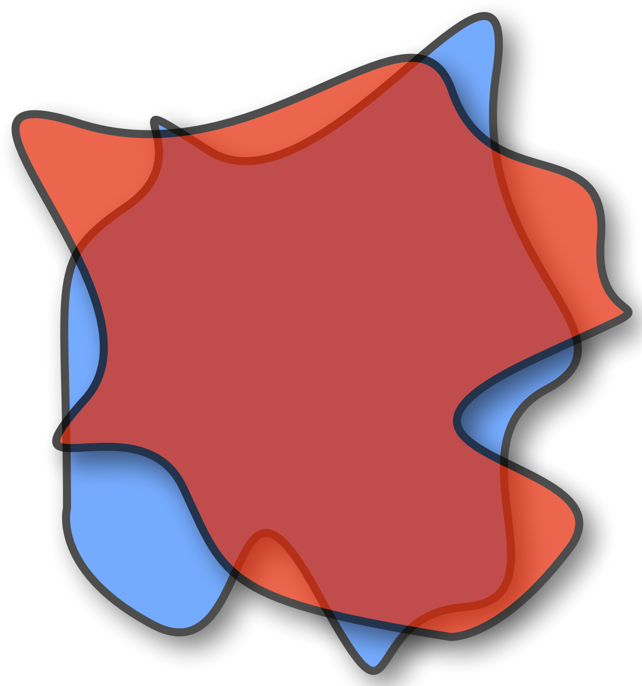

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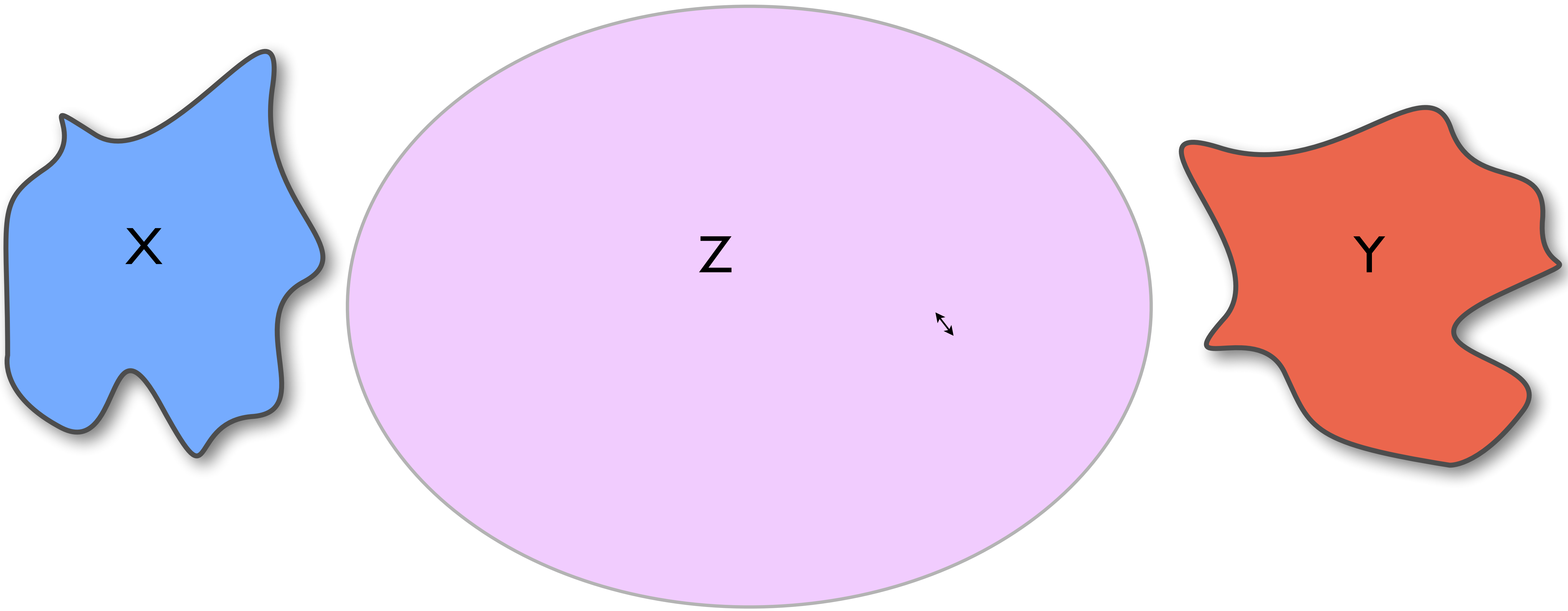

The Gromov–Hausdorff distance between X and Y is the smallest Hausdorff distance between all possible isometric embeddings of X and Y in a *same* metric space Z.

The Brownian tree

↝ Consequence of Aldous' theorem (Duquesne, Le Gall): there exists a compact metric space such that the convergence

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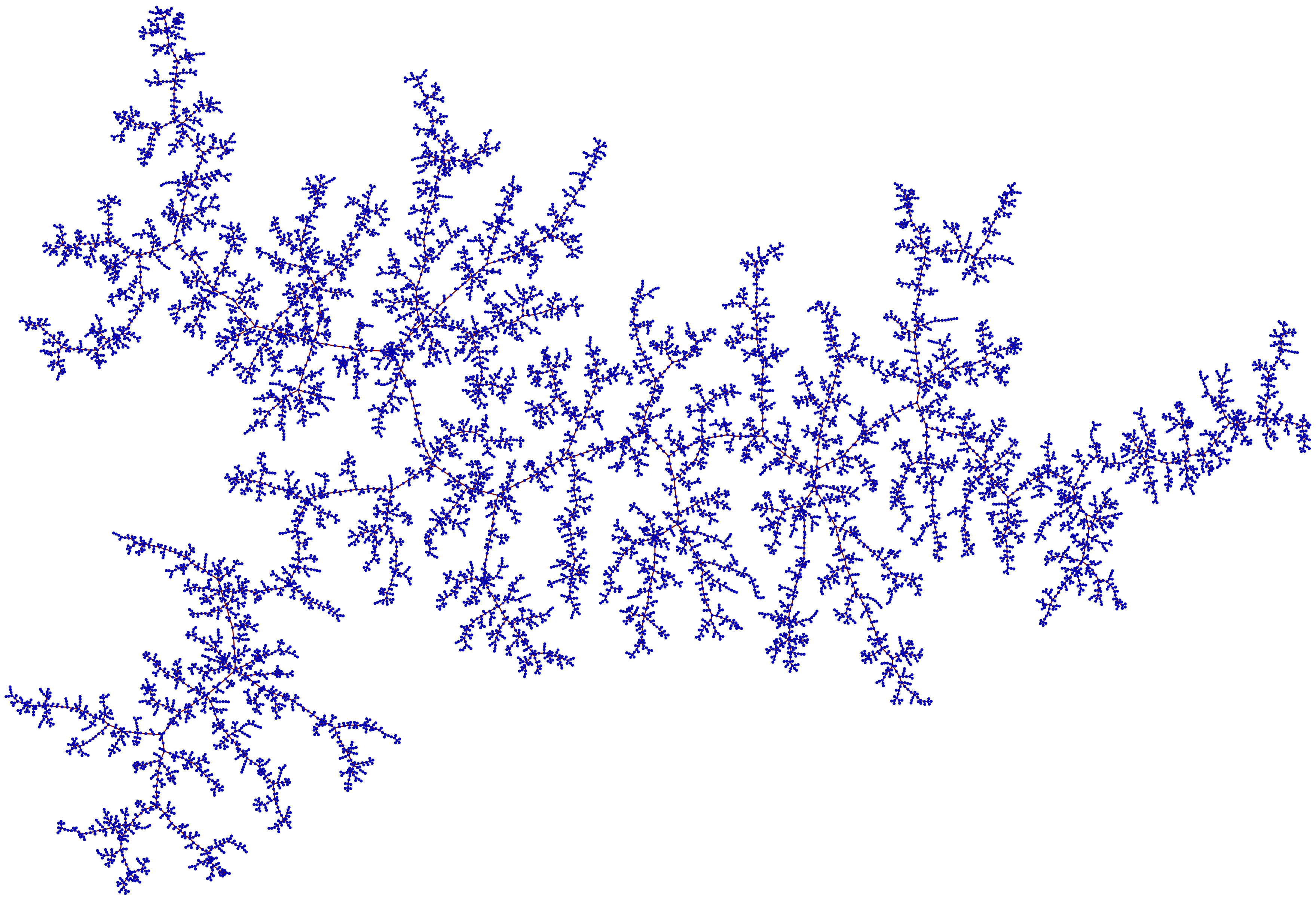
The metric space $\mathcal{T}_{\mathbb{e}}$ is called the *Brownian continuum random tree (CRT)*, and is coded by a Brownian excursion.



An approximation of a realization of a Brownian CRT

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⇝ Scaling limits are described use stable Lévy processes.

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WHAT ABOUT NON-CRITICAL OFFSPRING DISTRIBUTIONS?

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4. Because we condition on total population size, the distribution of $\mathscr{T}_n$ is unchanged by replacing ξ with another distribution χ in the same exponential family

$$P(\xi = i) = c\theta^i P(\chi = i), \qquad i \geq 0 \text{ for some } c, \theta.$$

Thus there is no essential loss of generality in considering only critical branching processes.

Condensation (subcritical case)

Let μ be a **subcritical** offspring distribution such that $\mu(n) \sim c/n^{1+\beta}$ with $\beta > 2$. Let $\mathcal{T}_n$ be a μ-Bienaymé tree conditioned on having n vertices.

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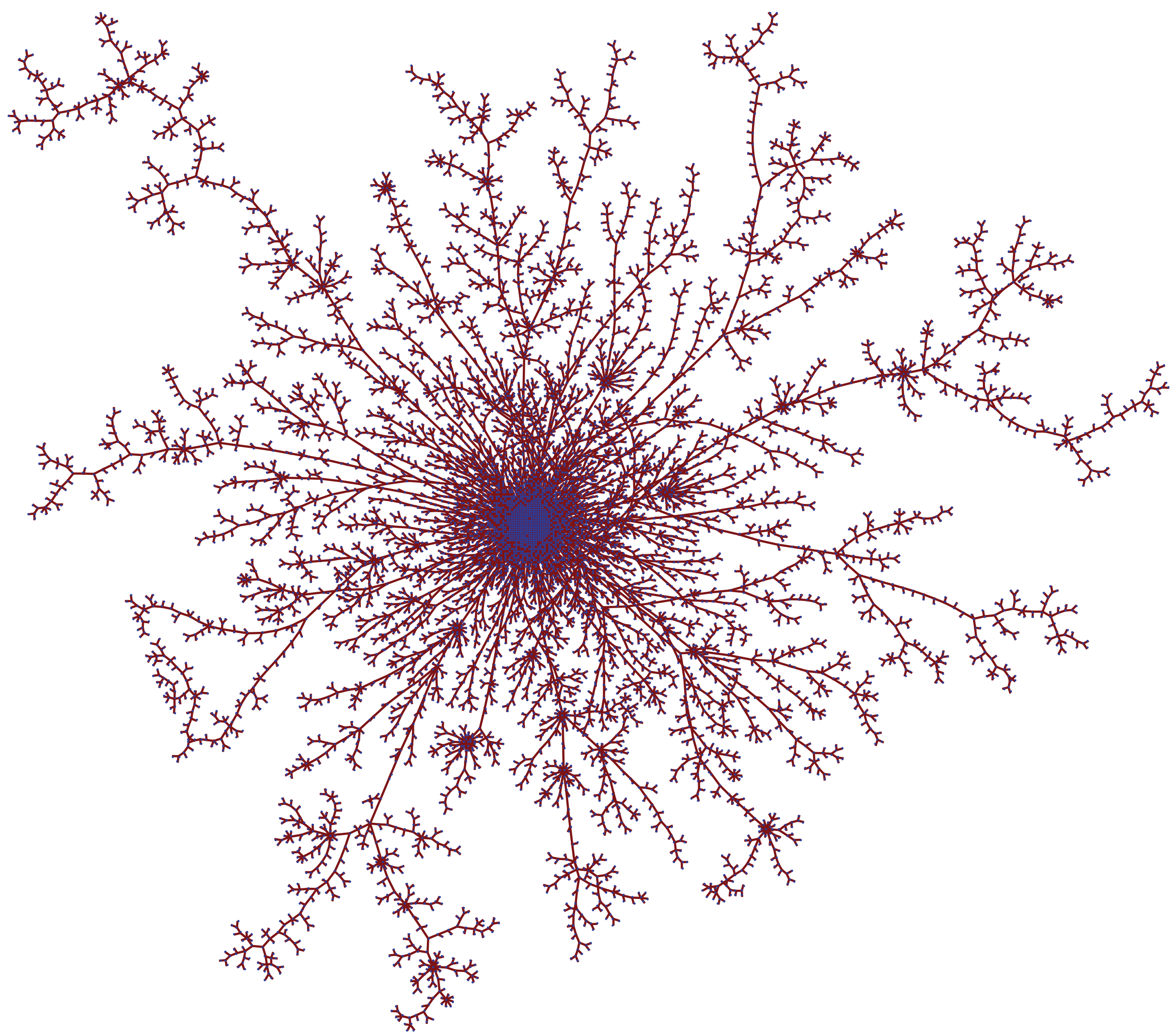

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Let m be the mean of μ. Denote by $\Delta(\mathcal{T}_n)$ the maximum degree of $\mathcal{T}_n$. Then

$$\frac{\Delta(\mathcal{T}_n)}{n} \quad \xrightarrow[n\to\infty]{(\mathbb{P})} \quad 1 - m.$$

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This is not true for any subcritical offspring distribution whose generating function has radius of convergence equal to 1 (even though there always is a local limit with a finite spine)!

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