DIFFERENTIAL CALCULUS

1. (a) $(x^8e^{-x^3} - x - 100)' = 8x^7e^{-x^3} - x^8 \cdot (3x^2e^{-x^3}) - 1 = x^7e^{-x^3}(8 - 3x^3) - 1$

(b)

$$\left(\frac{\ln(\sin^2(x))}{\cos(x)}\right)' = \frac{\frac{1}{\sin^2 x} \cdot (2\sin x\cos^2 x) + \ln(\sin^2 x)\sin x}{\cos^2 x} = \frac{2}{\sin x} + \frac{\ln(\sin^2 x)\sin x}{\cos^2 x}$$

(c) Set $y = \arctan(\sqrt{x})$ such that $\tan y = \sqrt{x}$. By implicitly differentiating with respect to x, we have

$$\sec^2 y \cdot y' = \frac{1}{2\sqrt{x}}.$$

In particular, since $\sec^2 y = 1 + \tan^2 y$ where $\sec y = \frac{1}{\cos y}$, we can write

$$y' = \arctan(\sqrt{x})' = \frac{1}{2\sqrt{x} \cdot \sec^2 y} = \frac{1}{2\sqrt{x}(1 + \tan^2 y)} = \frac{1}{2\sqrt{x}(1 + x)}.$$

2. Proceed by using the chain rule repeatedly:

(a)

$$f'(x) = e^{\sin(x^3+\cos x^2)}\left(\sin(x^3 + \cos x^2)\right)' = e^{\sin(x^3+\cos x^2)}\left(\cos(x^3 + \cos x^2)\right)(3x^2 - \sin x^2 \cdot 2x)$$

(b)

$$g'(x) = 2\cos\left(\frac{x^3+1}{x^2+1}\right) \cdot \left(\cos\left(\frac{x^3+1}{x^2+1}\right)\right)'$$
$$= -2\cos\left(\frac{x^3+1}{x^2+1}\right)\sin\left(\frac{x^3+1}{x^2+1}\right)\left((x^3+1)(x^2+1)^{-1}\right)'$$
$$= -2\cos\left(\frac{x^3+1}{x^2+1}\right)\sin\left(\frac{x^3+1}{x^2+1}\right) \cdot$$
$$\left((3x^2)\cdot(x^2+1)^{-1} - (x^3+1)\cdot(x^2+1)^{-2}\cdot(2x)\right).$$
$$= \frac{-x^4 - 3x^2 + 2x}{(x^2+1)^2}\sin\left(\frac{2(x^3+1)}{x^2+1}\right)$$

3. Recall that the derivative of a function at a point x_0 is precisely the slope of the tangent line at this point. Therefore, the graph of f has a horizontal tangent at points x_0 for which $f'(x_0) = 0$. We compute

$$f'(x) = (e^{\sin x + \cos x})' = e^{\sin x + \cos x}(\cos x - \sin x).$$

It follows that $f'(x) = 0$ exactly when $\cos x = \sin x$. In the main interval $[0, 2\pi)$, this happens at the points $x_1 = \frac{\pi}{4}$ and $x_2 = \frac{5\pi}{4}$. In $\mathbb{R}$, the solutions are given by $x = \frac{\pi}{4} + k\pi$ for any integer k.

4. The domain of $\ln(x)$ is $(0, \infty)$. Thus, the domain of $h(x) = \ln(\ln(x))$ is the set of points in $(0, \infty)$ such that $\ln(x) \in (0, \infty)$, i.e. $\mathrm{dom}(h) = (1, \infty)$.

The first derivative of h is given by

$$h'(x) = \frac{1}{x \ln x},$$

and so we compute the second derivative to be

$$\begin{aligned} h''(x) &= -\frac{(\ln x + 1)}{(x \ln x)^2} \\ &= -\left(\frac{1}{x^2 \ln x} + \frac{1}{(x \ln x)^2}\right) \\ &= \frac{-1}{x^2 \ln x}\left(1 + \frac{1}{\ln x}\right). \end{aligned}$$

Possible inflection points are where $h''(x) = 0$, i.e. when $\frac{1}{\ln x} = -1$. This has a solution for $x = \frac{1}{e}$. However, this value of x does not lie in the domain of h. Thus, there are no inflection points of h.

5. (a) From the definition of the derivative, we have

$$\begin{aligned} \frac{d}{dx}x^3 &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + 3h^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 \\ &= 3x^2. \end{aligned}$$

(b)

$$\begin{aligned}
\frac{d}{dx}(fg) &= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} g(x+h)\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h} \\
&= (\lim_{h\to 0} g(x+h))(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}) + f(x)(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}) \\
&= f'(x)g(x)+f(x)g'(x).
\end{aligned}$$

6. We rewrite $x^x = e^{x\ln x}$ and calculate the derivative as follows:

$$\begin{aligned}
\frac{d}{dx}(x^x) &= \frac{d}{dx}(e^{x\ln x}) \\
&= e^{x\ln x}\cdot\frac{d}{dx}(x\ln x) \\
&= e^{x\ln x}\cdot(\ln x+1) \\
&= x^x\cdot(\ln x+1).
\end{aligned}$$

7. Takeaway: What are the derivatives of sine and cosine? You will never forget...