Revision sheet solutions

I recommend trying to attempt the problems without referring to your lecture notes to begin with.

1. We differentiate $f'(x) = x^3 - 2x^2 = x^2(x-2)$. Thus f has critical points at $x = 0$ and $x = 2$. We have $f''(x) = 3x^2 - 4x$, so $f''(2) = 4 > 0$, meaning that $x = 2$ is a local minimum. However, $f''(0) = 0$, so we instead check the sign of f' on the respective sides of $x = 0$. We note that $f'(x) < 0$ for $0 < x < 2$ and $f'(x) < 0$ for $x < 0$, so this point is a stationary point which is neither a local min or local max.

   Since $f(x)$ tends to ∞ when $x \to \pm\infty$, the value $x = 2$ gives a global minimum, and there is no global maximum.

   A sketch shows a decreasing function as $x \to 0$ which becomes stationary at $x = 0$ but continues to decrease afterwards until $x = 2$, where the derivative changes sign and afterwards the function remains increasing.

   Since $f(2) = 2/3$ is the minimum value, f does not attain zero on $\mathbb{R}$. By the intermediate value theorem, since there are values of both sides of $x = 2$ such that $f(x) > 1$, there must be (at least) one value of x on each side of 2 such that $f(x) = 1$. If there are two such values on the same side, there must be a point where $f'(x) = 0$ in between (by MVT). However, as f is strictly decreasing on both sides of $x = 0$ where this is the only place where $f' = 0$, there cannot be any other point where $f(x) = 1$ and $x < 0$. Hence there are precisely two values of f where $f(x) = 1$.

2. (a) We use the partial fraction decomposition of the integrand

   $$\frac{5}{9(x-1)} + \frac{4}{3(x-1)^2} - \frac{5}{9(x+2)}.$$

   The integral thus becomes

   $$\frac{5}{9}\ln|x-1| - \frac{4}{3}(x-1)^{-1} - \frac{5}{9}\ln|x+2| + C.$$

   (b) Substitute $u = \sqrt{x}$ to obtain $\int_0^2 \log(u) 2u du$. Integrate by parts with $f = \log u$, $g' = 2u$ to get $[u^2 \log(u)]_0^2 - \int_0^2 u du = 4\log 2 - \frac{2^2}{2}$.

   (c) $\int \frac{d(\text{cabin})}{\text{cabin}} = \log|\text{cabin}| + \text{C}$

3. Use the chain rule and FTC to find $f'(x) = e^{x^4} \cdot 2x$.

4. The right hand side equals $\frac{1}{2} - \frac{\sqrt{3}i}{2}$, which in exponential form becomes $z^4 = e^{-\pi i/3}$. Taking 4th roots, we obtain $z = e^{-\pi i/12 + k\cdot\pi i/2}$ for $k = 0, 1, 2, 3$.

5. A normal at (x, y, z) is given by the gradient $(\partial_x F, \partial_y F, \partial_z F)$ where

   $$F = z^3 + y^3 + x^2 y - 2,$$

   i.e. $(2xy, 3y^2 + x^2, 3z^2)$ Evaluating at $(0, 1, 1)$ gives a normal $(0, 3, 3)$.

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The sphere $x^2+y^2+z^2=2$ has the same normal at $(0,1,1)$ and hence the same tangent plane, meaning that the sphere and the surface $F=0$ are tangent at this point.

6. (a) Multiply by the integrating factor $e^{\sqrt{x}}$ (or use variation of constants) to obtain $(ye^{\sqrt{x}})' = x$. Integrating gives $ye^{\sqrt{x}} = \frac{x^2}{2}+C$, i.e.

$$y = \left(\frac{x^2}{2}+C\right)e^{-\sqrt{x}}.$$

(b) We solve the characteristic equation $\lambda^2+\lambda+1=0$, giving $\lambda = \frac{-1}{2} \pm \frac{\sqrt{3}i}{2}$. The solutions to the homogeneous equation then becomes

$$y_h = Ae^{\frac{-1}{2}t}\cos(\frac{\sqrt{3}t}{2}) + Be^{\frac{-1}{2}t}\sin(\frac{\sqrt{3}t}{2}).$$

For the particular solution, we use the ansatz $y = at^4+bt^3+ct^2+dt+e$ to obtain

$$t^2 = y''+y'+y = 12at^2+6bt+2c+4at^3+3bt^2+2ct+d+at^4+bt^3+ct^2+dt+e,$$

giving $a=0$, $b=0$, $c=1$, $d=-2$, $e=0$.
Hence the general solution is $y = y_h + t^2 - 2t$.

7. Rewrite as $\dot{x} = \frac{3\dot{y}}{4}$, $\dot{y} = x - y$. Find the eigenvalues of the matrix

$$\begin{pmatrix} 0 & \frac{3}{4} \\ 1 & -1 \end{pmatrix}$$

as $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{-3}{2}$. We find two corresponding eigenvectors $v_1 = (3,2)$ and $v_2 = (-1,2)$. The general solution is given by $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1e^{t/2}v_1 + C_2e^{3t/2}v_2$.

8. We perform row operations on $\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ to make the LHS an identity matrix, giving the inverse $A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 & -2 \\ 2 & -1 & 1 \\ -1 & -2 & 7 \end{pmatrix}$.