DIFFERENTIAL CALCULUS

1. (a) We have to solve the equation $f(x) = g(x)$ and compute that

$$\begin{aligned} & f(x) = g(x) \\ \Longleftrightarrow\ & 4x^3 + 2x^2 - 5x - 2 = 2x^2 - x - 2 \\ \Longleftrightarrow\ & 4x^3 - 4x = 0 \\ \Longleftrightarrow\ & 4x(x^2 - 1) = 0 \\ \Longleftrightarrow\ & 4x(x+1)(x-1) = 0 \\ \Longleftrightarrow\ & (x+1)x(x-1) = 0. \end{aligned}$$

Therefore, we get as result

$$x_1 = -1,\ x_2 = 0,\ x_3 = 1.$$

(b) The local minimum and the local maximum must satisfy the conditions

$$\begin{aligned} f'(x_{\min}) = 0 \ \text{ and } \ f''(x_{\min}) > 0, \\ f'(x_{\max}) = 0 \ \text{ and } \ f''(x_{\max}) < 0. \end{aligned}$$

Therefore, we must solve the equation $f'(x) = 0$ and calculate

$$\begin{aligned} f'(x) &= (4x^3 + 2x^2 - 5x - 2)' \\ &= 4(x^3)' + 2(x^2)' - 5x' - 2' \\ &= 4 \cdot 3x^2 + 2 \cdot 2x - 5 - 0 \\ &= 12x^2 + 4x - 5 \\ &= (2x - 1) \cdot (6x + 5) \\ &= 12(x - \tfrac{1}{2}) \cdot (x + \tfrac{5}{6}) = 0. \end{aligned}$$

Thus, we obtain that $x_{\min}, x_{\max} \in \left\{\frac{1}{2}, -\frac{5}{6}\right\}$.

To decide, which point is the minimum and which point is the maximum, we look at the conditions for the second derivative $f''(x)$ and compute

$$\begin{aligned} f''(x) &= (f'(x))' \\ &= (12x^2 + 4x - 5)' \\ &= 12(x^2)' + 4x' - 5' \\ &= 12 \cdot 2x + 4 \\ &= 24x + 4 \\ &= 4(6x + 1), \end{aligned}$$

to see that

$$f''(\tfrac{1}{2}) = 4(6\tfrac{1}{2} + 1) = 4 \cdot 4 = 16 > 0,$$
$$f''(-\tfrac{5}{6}) = 4(6(-\tfrac{5}{6}) + 1) = 4 \cdot (-4) = -16 < 0.$$

Therefore, we conclude that $x_{\min} = \frac{1}{2}$ and $x_{\max} = -\frac{5}{6}$.

2. We first find the critical points (the local extrema), that is, the points x for which $f'(x) = 0$. The derivative is

$$\begin{aligned} f'(x) &= (x^4 - 4x^3 + 4x^2 - 3)' \\ &= (x^4)' - 4(x^3)' + 4(x^2)' - 3' \\ &= 4x^3 - 4 \cdot 3x^2 + 4 \cdot 2x - 0 \\ &= 4x^3 - 12x^2 + 8x \\ &= 4x(x^2 - 3x + 2) \\ &= 4x(x-1)(x-2). \end{aligned}$$

Hence the local extrema occur at $x_1 = 0$, $x_2 = 1$ and $x_3 = 2$.
To determine whether the f-values at these three points are local minima or local maxima, we look at the second derivative, which is given by

$$\begin{aligned} f''(x) &= (f'(x))' \\ &= (4x^3 - 12x^2 + 8x)' \\ &= 4(x^3)' - 12(x^2)' + 8x' \\ &= 4 \cdot 3x^2 - 12 \cdot 2x + 8 \\ &= 12x^2 - 24x + 8 \\ &= 4(3x^2 - 6x + 2). \end{aligned}$$

Because we have that

$$\begin{aligned} f''(0) &= 4(3 \cdot 0^2 - 6 \cdot 0 + 2) = 8 > 0, \\ f''(1) &= 4(3 \cdot 1^2 - 6 \cdot 1 + 2) = -4 < 0, \\ f''(2) &= 4(3 \cdot 2^2 - 6 \cdot 2 + 2) = 8 > 0, \end{aligned}$$

we have that $x = 0$ and $x = 2$ correspond to local minima and that $x = 1$ corresponds to a local maximum.

To find global extrema on the interval $[-2, 3]$, we compare the values of f at the points $0, 1, 2$ with the two endpoints $e_1 = -2$ and $e_2 = 3$ of the interval $[-2, 3]$, over which the global extrema should be identified.
Putting everything together, we have that

$$\begin{aligned} f(-2) &= (-2)^4 - 4 \cdot (-2)^3 + 4 \cdot (-2)^2 - 3 = 16 + 32 + 16 - 3 = 61, \\ f(0) &= 0^4 - 4 \cdot 0^3 + 4 \cdot 0^2 - 3 = -3, \\ f(1) &= 1^4 - 4 \cdot 1^3 + 4 \cdot 1^2 - 3 = 1 - 4 + 4 - 3 = -2, \\ f(2) &= 2^4 - 4 \cdot 2^3 + 4 \cdot 2^2 - 3 = 16 - 32 + 16 - 3 = -3, \\ f(3) &= 3^4 - 4 \cdot 3^3 + 4 \cdot 3^2 - 3 = 81 - 108 + 36 - 3 = 6. \end{aligned}$$

Hence the global maximum over the interval $[-2,3]$ of the function $f(x) = x^4 - 4x^3 + 4x^2 - 3$ is 61 and the global minimum is -3.
These global extrema occur at the points $x_{\max} = -2$, $x_{\min}^{(1)} = 0$ and $x_{\min}^{(2)} = 2$, because we have that $f(-2) = 61$ and that $f(0) = f(2) = -3$.

3. Using the product rule and the chain rule, we compute that

$$\begin{aligned} y' &= y'(x) \\ &= \left[(x^4-1)^3 \ln(x+1)\right]' \\ &= ((x^4-1)^3)' \ln(x+1) + (x^4-1)^3 (\ln(x+1))' \\ &= 3(x^4-1)^2 \cdot (x^4-1)' \ln(x+1) + (x^4-1)^3 \frac{1}{x+1}(x+1)' \\ &= 3(x^4-1)^2(4x^3-0)\ln(x+1) + (x^4-1)^3\frac{1}{x+1}\cdot 1 \\ &= 12x^3(x^4-1)^2\ln(x+1) + \frac{(x^4-1)^3}{x+1}. \end{aligned}$$

So we have that

$$y'(0) = 12\cdot 0^3(0^4-1)^2\ln(0+1) + \frac{(0^4-1)^3}{0+1} = 0 + \frac{(-1)^3}{1} = (-1)^3 = -1.$$

Therefore, the tangent t to the curve y is given by $t(x) = -x$, because $t(0) = 0 = y(0)$ and $t'(0) = -1 = y'(0)$.
The normal $n(x) = mx + b$ to the curve at the origin $(x,y) = (x, y(x)) = (0,0)$ must have a coefficient of direction $m = -\frac{1}{-1} = 1$ to make an angle of $90°$ between the lines $t(x) = -x$ and $n(x) = x + b$, so the normal is $n(x) = x + b$ for some $b \in \mathbb{R}$. Because this normal passes through the origin $(x,y) = (x,y(x)) = (0, y(0)) = (0, n(0)) = (0,0)$, the equation for the normal must be $n(x) = x$.

4. a) By the Mean Value Theorem, $f(1) - f(0) = f'(\varepsilon)$ for some $\varepsilon \in [0,1]$. Since $f'(\varepsilon) \le 2$ by assumption, it follows that $f(1) \le 2 + f(0) = 1$. Equality holds for the function $f(x) = -1 + 2x$.

   b) Let $g(x) = f(x) - x$. We have $g(0) = f(0) - 0 \ge 0$ and $g(1) = f(1) - 1 \le 0$. If $f(0) = 0$ or $f(1) = 1$, then 0 resp. 1 is a fixed point and we are done. Otherwise, the conditions of the Intermediate Value Theorem apply, meaning that there is a value $x \in (0,1)$ such that $g(x) = f(x) - x = 0$. This value of x is a fixed point of f.

5. We have $f'(x) = (x-1)^3(5x-1)$, having zeros at $(x, f(x)) = (1,0), (1/5, 2^8/5^5)$. We calculate the second derivative as $f''(x) = 3(x-1)^2(5x-1) + 5(x-1)^3$.

   For $x = 1/5$, we have $f''(x) = -64/25 < 0$ so this is a local maximum.

Near $x = 1$, e.g. in the intervals $x \in (0.5, 0)$ and $x \in (0, 1.5)$, the function $f(x)$ is strictly greater than $f(0) = 0$, so f has a local minimum at $x = 1$ (despite the second derivative being zero!)

There are no global extrema since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

For the sketch, see eg. GeoGebra (you should have taken into account the zeros of f, the local extrema and the behaviour of f as x tends to plus/minus infinity).