DIFFERENTIAL CALCULUS

1. (a) Can you see directly why this integral is 0?

   (b)
$$\int e^{-7x}\,\mathrm{d}x = \frac{-1}{7}e^{-7x} + C.$$

   (c)
$$\int \sqrt{5x}\,\mathrm{d}x = \sqrt{5}\int x^{1/2}\,\mathrm{d}x = \sqrt{5}\cdot\frac{2x^{3/2}}{3} + C.$$

   (d)
$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = -2e^{-\sqrt{x}}\Big|_{x=0}^{x=\infty} = 2.$$

   (e)
$$\int_2^8 \frac{1}{x}\,\mathrm{d}x = \ln|x|\Big|_{x=2}^{x=8} = \ln 8 - \ln 2 = \ln 4.$$

   (f)
$$\int \mathrm{d}x = \int 1\,\mathrm{d}x = x + C.$$

2. Recall the formula
$$\int f(x)\cdot g'(x)\,\mathrm{d}x = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,\mathrm{d}x.$$

   (a) For $f(x) = \ln(\sin x)$ and $g'(x) = \cos x$, we compute
$$\begin{aligned}\int \cos x \ln(\sin x)\,\mathrm{d}x &= \ln(\sin x)\cdot\sin x - \int \frac{\cos x}{\sin x}\cdot\sin x\,\mathrm{d}x\\ &= \ln(\sin x)\cdot\sin x - \sin x + C\\ &= \sin x\cdot(\ln(\sin x) - 1) + C.\end{aligned}$$

   (b) Set $f(x) = x$ and $g'(x) = \frac{1}{\cos^2 x}$;
$$\begin{aligned}\int \frac{x}{\cos^2 x}\,\mathrm{d}x &= x\tan x - \int \tan x\,\mathrm{d}x\\ &= x\tan x + \ln|\cos x| + C.\end{aligned}$$

   (c) Set $f_1(x) = x^3$ and $g_2'(x) = e^x$;
$$\int x^3 e^x\,\mathrm{d}x = x^3 e^x - 3\int x^2 e^x\,\mathrm{d}x.$$

Now integrate by parts again with $f_2(x) = x^2$ and $g_2'(x) = e^x$;

$$\int 3x^2 e^x \,\mathrm{d}x = x^2 e^x - 2\int x e^x \,\mathrm{d}x.$$

We also solve the last integral by parts with $f_3(x) = x$ and $g_3'(x) = e^x$;

$$\int x e^x \,\mathrm{d}x = x e^x - \int e^x \,\mathrm{d}x = x e^x - e^x + C.$$

The final result is therefore

$$x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C.$$

(d) Set $f(x) = \ln(x^2+1)$ and $g'(x) = 1$;

$$\begin{aligned}\int \ln(x^2+1)\,\mathrm{d}x &= \int 1 \cdot \ln(x^2+1)\,\mathrm{d}x \\ &= x\ln(x^2+1) - \int x \cdot \frac{2x}{x^2+1}\,\mathrm{d}x \\ &= x\ln(x^2+1) - 2\int \frac{(x^2+1)-1}{x^2+1}\,\mathrm{d}x \\ &= x\ln(x^2+1) - 2\int 1 - \frac{1}{x^2+1}\,\mathrm{d}x \\ &= x\ln(x^2+1) - 2x + 2\arctan(x) + C.\end{aligned}$$

(e) Proceed with $f(x) = \ln(x)$ and $g'(x) = x$;

$$\begin{aligned}\int x\ln x\,\mathrm{d}x &= \frac{x^2}{2}\ln(x) - \frac{1}{2}\int \frac{x^2}{x}\,\mathrm{d}x \\ &= \frac{x^2}{2}\ln(x) - \frac{1}{4}x^2 + C.\end{aligned}$$

(f) For $f(x) = \sin x$ and $g'(x) = \sin x$;

$$\int \sin^2 x\,\mathrm{d}x = -\sin x\cos x + \int \cos^2 x\,\mathrm{d}x,$$

thus

$$\int \sin^2 x - \int \cos^2 x\,\mathrm{d}x = -\sin x \cos x.$$

Since $\int \sin^2 x + \int \cos^2 x\,\mathrm{d}x = x + C_1$, we can sum the latter two expressions to conclude that

$$\int \sin^2 x\,\mathrm{d}x = \frac{1}{2}(x - \sin x\cos x) + C.$$

Bonus: We have $\int_0^{\pi/2} \sin^2 x dx = \int_0^{\pi/2} \cos^2 x dx$ (since $\cos(x) = \sin(x+\pi/2)$), and adding these give $\pi/2$ by the identity $\sin^2 x + \cos^2 x = 1$. Thus, we have $\int_0^{\pi/2} \sin^2 x dx = \pi/4$. (Alternatively, we can use the half-angle formula.)

3. (a) The two graphs intersect when $f(x) = g(x)$:
$$4x^3 + 2x^2 - 5x - 2 = 2x^2 - x - 2$$
simplifies to
$$(x+1)x(x-1) = 0,$$
and we therefore deduce that $x_1 = -1, x_2 = 0, x_3 = 1$.

(b)
$$\begin{aligned}\int_{x_1}^{x_3} \big(f(x) - g(x)\big)\,\mathrm{d}x &= \int_{-1}^{1} (4x^3 - 4x)\,\mathrm{d}x \\ &= x^4 - 2x^2 \Big|_{x=-1}^{x=1} \\ &= 0.\end{aligned}$$

(c) Where the graph of f is above g, integrate $f(x)-g(x)$. Otherwise, integrate $g(x) - f(x)$. The area A of the shaded region is therefore
$$\begin{aligned}A &= \int_{-1}^{0} (f(x) - g(x))\,\mathrm{d}x + \int_{0}^{1} (g(x) - f(x))\,\mathrm{d}x \\ &= \int_{-1}^{0} (4x^3 - 4x)\,\mathrm{d}x + \int_{0}^{1} (4x - 4x^3)\,\mathrm{d}x \\ &= x^4 - 2x^2 \Big|_{x=-1}^{x=0} + 2x^2 - x^4 \Big|_{x=0}^{x=1} = 2.\end{aligned}$$

4. Consider a value $x \in (0,1)$ where $f(x) \neq 0$; assume WLOG that $f(x) = c > 0$. Take a small interval $[a, b]$ around this point where $f(x) > c/2$ (this is possible, by continuity!) and a function $g(x)$ which is zero on $[0, a]$ and $[b, 1]$ but positive inside (a, b) and greater than 1 on some interval (c, d) where $a < c$ and $d < b$ (think why such a function exists!). Then, the integral $\int_0^1 f(x)g(x)dx = \int_a^b f(x)g(x)dx \geq \int_c^d f(x)g(x)dx \geq \int_c^d c/2 \cdot 1dx > 0$, a contradiction. A similar argument deals with the cases $x = 0, 1$.

5. Remember that
$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{g(x)}^{h(x)} f(t)\,\mathrm{d}t = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$$
by the chain rule and the second fundamental theorem of calculus. In particular,
$$f'(x) = \frac{2\sin(2x)}{2x} - \frac{\sin x}{x}.$$
Local extrema occurs in points x for which $f'(x) = 0$, namely when $\sin 2x = \sin x$. By the double angle formula, $\sin 2x = 2 \sin x \cos x$, thus the above condition is satisfied for $\sin x = 0$, or for $\cos x = \frac{1}{2}$. In our range, this occurs when $x = \pi$, or $x = \frac{\pi}{3}$.
Using the second derivative test, we see that $f''(\pi) > 0$, and $f''(\frac{\pi}{3}) < 0$, thus only $x = \frac{\pi}{3}$ gives a local maximum.