4.1. Conservation laws and critical times Consider the PDE

$$u_y + \partial_x(f(u)) = 0\,.$$

In the following cases, compute the critical time y_c (i.e., the first time when the solution becomes nonsmooth):

(a) $f(u) = \frac{1}{2}u^2$, the initial datum is $u(x,0) = \sin(x)$.

(b) $f(u) = \sin(u)$, the initial datum is $u(x,0) = x^2$.

(c) $f(u) = e^u$, the initial datum is $u(x,0) = x^3$.

4.2. Weak solutions Consider the PDE

$$\partial_y u + \partial_x\Big(\frac{u^4}{4}\Big) = 0$$

in the region $x \in \mathbb{R}$ and $y > 0$.

(a) Show that the function $u(x,y) := \sqrt[3]{\frac{x}{y}}$ is a classical solution of the PDE.

(b) Show that the function

$$u(x,y) := \begin{cases} 0 & \text{if } x > 0, \\ \sqrt[3]{\frac{x}{y}} & \text{if } x \le 0. \end{cases}$$

is a *weak* solution of the PDE.