5.1. Finding shock waves

Consider the transport equation

$$u_y + u^2 u_x = 0,$$

with initial condition $u(x,0) = 1$ for $x \leq 0$, $u(x,0) = 0$ for $x \geq 1$, and

$$u(x,0) = \sqrt{1-x} \quad \text{for} \quad 0 < x < 1.$$

(a) Find the solution using the method of characteristics. Up to which time is the solution defined in a classical sense?

(b) Find a weak solution for all times $y \geq 0$.

5.2. Weak solutions

Consider the transport equation

$$u_y + \frac{1}{2}\partial_x \left(u^2\right) = 0. \tag{1}$$

(a) Suppose that u is a classical solution to the previous transport equation. What equation does u^2 fulfil? Write it in the form

$$v_y + \partial_x \left(F(v)\right) = 0, \tag{2}$$

for some appropriate F.

(b) Consider the weak solution of Equation (1) given by

$$w(x,y) = \begin{cases} 3 & \text{if } x < \frac{3}{2}y - 1 \\ 0 & \text{if } x > \frac{3}{2}y - 1. \end{cases}$$

Show that w^2 is not a weak solution of (2). Can you explain what is the problem?

5.3. Weak solutions II

Consider the equation

$$e^{-u}u_x + u_y = 0,$$

with initial value $u(x,0) = 0$ if $x < 0$, and $u(x,0) = \alpha > 0$ if $x > 0$.

(a) Find a weak solution for any $\alpha > 0$ with a single discontinuity for $y \geq 0$.

(b) Show that such solution fulfils the entropy condition for all $\alpha > 0$.

5.4. Multiple Choice Determine the correct answer(s) to each point.

(a) The second order linear PDE given by

$$u_x + x^2 u_{xx} + 2x\sin(y)u_{xy} - \cos^2(y)u_{yy} + e^x = 0,$$

is

- ☐ Everywhere hyperbolic
- ☐ Parabolic for $\{y : \cos(y) = 0 \text{ or } \sin(y) = 0\} = \{k\frac{\pi}{2} :\in \mathbb{Z}\}$ and hyperbolic elsewhere
- ☐ Parabolic in $x = 0$, and hyperbolic elsewhere

(b) The following conservation law

$$\begin{cases} u_y + f(u)_x = 0, \\ u(x,0) = c > 0 \text{ for } \{x < 0\} \text{ and } u(x,0) = 0 \text{ for } \{x \geq 0\} \end{cases}$$

has a shock curve of slope equal to 8 if

- ☐ $c = 2$ and $f(u) = u^4$
- ☐ $c = 2$ and $f(u) = -u^4$
- ☐ $c = 1$ and $f(u) = 2u^2 + 6u - 1$