6.1. Wave equation

Let $c > 0$, and consider the wave equation posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = \sin(x), & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

(a) Solve the Cauchy problem. Identify the forward and the backward wave, and express the solution with separated variables, that is, $u(x,t) = v(x)w(t)$ for some functions v and w. (*Hint: Recall that* $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$.)

(b) Show that u is $\frac{2\pi}{c}$-periodic in the t variable. That is, show that $u(x,t) = u(x, t+T)$ where $T = \frac{2\pi}{c}$.

6.2. Odd initial data

Consider the general wave equation posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Suppose that both f and g are odd functions (that is, $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}$). Show that the solution u is also an odd function in x, for each time $t > 0$ (that is, $u(-x,t) = -u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$).

(*Remark: An analogous result can be proved for even functions.*)

6.3. Zero boundary condition

Use the previous exercise to solve the following Cauchy problem posed for $x > 0$ and $t > 0$, with zero boundary condition at $x = 0$,

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x,t) \in (0,\infty) \times (0,\infty), \\ u(0,t) = 0, & t \in (0,\infty), \\ u(x,0) = x^2, & x \in (0,\infty), \\ u_t(x,0) = 0, & x \in (0,\infty). \end{cases}$$

6.4. Time reversible

Consider the Cauchy problem posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= f(x), & x \in \mathbb{R}, \\ u_t(x,0) &= g(x), & x \in \mathbb{R}. \end{cases}$$

Let $\tilde{u}(x,t) := u(x,-t)$. Show that $\tilde{u}(x,t)$ solves the Cauchy problem posed for $-\infty < x < \infty$ and $t < 0$,

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} &= 0, & (x,t) \in \mathbb{R} \times (-\infty,0), \\ \tilde{u}(x,0) &= f(x), & x \in \mathbb{R}, \\ \tilde{u}_t(x,0) &= -g(x), & x \in \mathbb{R}. \end{cases}$$

That is, we are showing that the wave equation is reversible in time. If a function solves a wave equation, the same function with time reversed also solves a the wave equation with the same initial condition and opposite initial velocity.

6.5. Multiple Choice Determine the correct answer.

(a) Consider the one dimensional wave equation given by

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = \arctan(x), & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Then, the asymptotic value of the solution at any $\bar{x} \in \mathbb{R}$ (i.e. $\lim_{t\to+\infty} u(\bar{x},t)$) is equal to

- ☐ 0
- ☐ $\pi/2$
- ☐ $\pi/2c$

(b) Given

$$\begin{cases} u_{tt} - \pi^2 u_{xx} = 0, \\ u(x,0) = x^2, & x \in \mathbb{R}, \\ u_t(x,0) = -\sin(x), & x \in \mathbb{R}. \end{cases}$$

the value of u at the point $(x,t) = (\pi, 2)$ is equal to

- ☐ 0
- ☐ $5\pi^2$
- ☐ $3\pi^2$