7.1. Nonhomogeneous wave equation Solve the following initial value problem:

$$\begin{cases} u_{tt} - u_{xx} = 1, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = 1, & x \in \mathbb{R}, \\ u_t(x,0) = 1, & x \in \mathbb{R}. \end{cases}$$

7.2. Strange wave equation Show that the following partial differential equation admits a solution

$$\begin{cases} u_{tt} - u_{xx} = \frac{u_t^2 - u_x^2}{2u}, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = x^4, & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Hint: Consider the function $v(x,t) = \sqrt{u(x,t)}$. What equation does it satisfy?

7.3. Symmetries Let $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be a solution of the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t), & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

By means of the uniqueness of the solution of the wave equation, show that

(a) If f, g, and F are spatially odd (odd with respect to x), then u is also spatially odd. (That is, $f(-x) = -f(x)$, $g(-x) = -g(x)$ and $F(-x,t) = -F(x,t)$ imply $u(-x,t) = -u(x,t)$.)

Hint: Consider the function $v(x,t) = -u(-x,t)$. What equation does it satisfy?

(b) If f, g, and F are spatially even (even with respect to x), then u is also spatially even. (That is, $f(-x) = f(x)$, $g(-x) = g(x)$ and $F(-x,t) = F(x,t)$ imply $u(-x,t) = u(x,t)$.)

(c) If f, g, and F are L-periodic, then u is also L-periodic. (That is, $f(x) = f(x+L)$, $g(x) = g(x+L)$ and $F(x,t) = F(x+L,t)$, imply $u(x,t) = u(x+L,t)$.)

7.4. Wave equation on a ring Let $u : [0,1] \times [0,\infty) \to \mathbb{R}$ be a solution of the wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x,t) \in [0,1] \times (0,\infty), \\ u(x,0) = x - x^2, & x \in [0,1], \\ u_t(x,0) = 0, & x \in [0,1], \\ u(0,t) = u(1,t), & t \in (0,\infty), \\ u_x(0,t) = u_x(1,t), & t \in (0,\infty). \end{cases}$$

Compute $u(\frac{1}{2}, 2021)$.

7.5. Multiple Choice Determine the correct answer.

(a) Consider the modified one dimensional wave equation of Problem 6.5 with nonhomogeneous right hand side

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(t), & \\ u(x,0) = \arctan(x), & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Suppose also that there exists a function $m = m(t) \in C^2$ such that $m_{tt} = F$ and $m(0) = m_t(0) = 0$. Then, without applying directly the d'Alembert formula (but possibly using Problem 6.5), the asymptotic value of u as $t \to \infty$, i.e. $\lim_{t\to+\infty} u(\bar{x}, t)$, is well defined and finite for all $\bar{x} \in \mathbb{R}$

- ☐ always because it is equal to 0.
- ☐ if $\lim_{t\to+\infty} m(t)$ exists and it is finite.
- ☐ if $\lim_{t\to+\infty} m'(t)$ exists and it is finite.