9.1. Separation of variables for non-homogeneous problems

Solve the following equations using the method of separation of variables and superposition principle. If the boundary conditions are non-homogeneous, find a suitable function satisfying the boundary conditions, and subtract it from the solution.

(a)

$$\begin{cases} u_t - u_{xx} &= t + 2\cos(2x), & (x,t) \in (0,\pi/2) \times (0,\infty), \\ u_x(0,t) &= 0, & t \in (0,\infty), \\ u_x(\pi/2,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= 1 + 2\cos(6x), & x \in [0,\pi/2]. \end{cases}$$

(b)

$$\begin{cases} u_t - u_{xx} &= 1 + x\cos(t), & (x,t) \in (0,1) \times (0,\infty), \\ u_x(0,t) &= \sin(t), & t \in (0,\infty), \\ u_x(1,t) &= \sin(t), & t \in (0,\infty), \\ u(x,0) &= 1 + \cos(2\pi x), & x \in [0,1]. \end{cases}$$

Hint: The function $w(x,t) = x\sin(t)$ fulfills the boundary conditions from above.

(c) Mixed Boundary Conditions.

$$\begin{cases} u_t - u_{xx} &= \sin(9x/2), & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u_x(\pi,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= \sin(3x/2), & x \in [0,\pi]. \end{cases}$$

(d)

$$\begin{cases} u_t - u_{xx} &= -u, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u(\pi,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= \sin(x), & x \in [0,\pi]. \end{cases}$$