11.1. Uniqueness of solution

Let $D \subseteq \mathbb{R}^2$ be a planar domain. Let $f : \partial D \to \mathbb{R}$ be a continuous function defined on the boundary.

Show that the following elliptic problem

$$\begin{cases} \Delta u &= u, \quad \text{in } D, \\ u &= f, \quad \text{on } \partial D, \end{cases}$$

admits at most one smooth solution.

Hint: suppose that u_1 and u_2 are two solutions of the above problem. What can we say about $u_1 - u_2$?

11.2. Heat equation

Consider a smooth solution $u : [0,1] \times [0,\infty) \to \mathbb{R}$ of the heat equation

$$\begin{cases} u_t - u_{xx} &= 0, \quad (x,t) \in (0,1) \times (0,\infty), \\ u(x,0) &= x(1-x), \quad 0 \leq x \leq 1, \\ u(0,t) = u(1,t) &= 0, \quad t \in (0,\infty). \end{cases}$$

Show that $0 \leq u(0.5, 100) \leq 0.00001$.

Hint: Notice that the function $w := e^{-\pi^2 t} \sin(\pi x)$ solves the same heat equation with a different initial condition.

11.3. Separation of variables for elliptic equations

(a) Find the solution to

$$\begin{cases} \Delta u &= 0, \quad \text{for } 0 < x < \pi, 0 < y < \pi, \\ u(x,0) = u(x,\pi) &= 0, \quad \text{for } 0 \leq x \leq \pi, \\ u(0,y) &= 0, \quad \text{for } 0 \leq y \leq \pi \\ u(\pi,y) &= \sin(y), \quad \text{for } 0 \leq y \leq \pi. \end{cases}$$

(b) Find the solution to

$$\begin{cases} \Delta u &= \sin(x) + \sin(2y), \quad \text{for } \pi < x < 2\pi, \pi < y < 2\pi, \\ u(x,\pi) &= 0, \quad \text{for } \pi \leq x \leq 2\pi, \\ u(x,2\pi) &= -\sin(x), \quad \text{for } \pi \leq x \leq 2\pi, \\ u(\pi,y) &= 0, \quad \text{for } \pi \leq y \leq 2\pi, \\ u(2\pi,y) &= -\frac{\sin(2y)}{4}, \quad \text{for } \pi \leq y \leq 2\pi. \end{cases}$$

Hint: Find a simple function $f(x,y)$ such that $v := u + f(x,y)$ is harmonic (i.e. $\Delta v = 0$). Then, solve for v.