1.1. Classification of PDEs Determine the order of the following PDEs. Determine also whether they are linear or not. If they are linear, determine if they are homogeneous or not, and if they are not linear, determine if they are quasilinear.

(a) $\Delta(\Delta u) = 5u$.

Fourth order. Linear. Homogeneous.

(b) $10^{20}u + \sin(u_x) = u_{xx}$

Second order. Not linear. Quasilinear.

(c) $e^{\Delta u} = u$.

The PDE can be rewritten as $\Delta u = \log(u)$. Second order. Not linear. Quasilinear.

(d) $\partial_x(uu_y) = \partial_y(uu_x)$.

Notice that $\partial_x(uu_y) = u_x u_y + uu_{xy}$ and $\partial_y(uu_x) = u_y u_x + uu_{xy}$. So the PDE is always true (at least for smooth functions) and should not be categorized as linear/not linear or homogeneous/not homogenous.

1.2. Solutions to PDEs Check whether each of the following PDEs has a solution u that is a polynomial and, if it exists, determine a polynomial that solves the PDE.

(a) $\Delta u = x + y$.

The polynomial $u(x, y) = \frac{1}{6}x^3 + \frac{1}{6}y^3$ solves the PDE.

(b) $u_{xx} = -u$, with $u(0) = 1$.

The general solution of the ODE $u_{xx} = -u$ is $u(x) = \alpha \sin(x) + \beta \cos(x)$. Hence, since $u(0) = 1$, it holds $u(x) = \alpha \sin(x) + \cos(x)$, which is not a polynomial for any choice of $\alpha \in \mathbb{R}$.

(c) $u_{xx} + u_{xy} = \sin(x)$.

If u is a polynomial, then $u_{xx} + u_{xy}$ is a polynomial. Therefore, since $\sin(x)$ is not a polynomial, there is not a solution u which is a polynomial.

(d) $u_{xyx}^2 + u_{yxy} = e^u$.

If u is a polynomial then $u_{xyx}^2 + u_{yxy}$ is a polynomial. Hence, if u solves and is a polynomial, then e^u is a polynomial. But if u and e^u are both polynomial then u must be constant (think of the growth as x goes to infinity). Since $u \equiv \text{const}$ is not a solution of the PDE, there PDE does not have a polynomial solution.

(e) $u_{xx} + u_y + u_{xy} = x^2y$.

The polynomial $u(x, y) = \frac{1}{2}x^2y^2 - \frac{y^3}{3} - xy^2 + y^2$ solves the PDE.

1.3. Solutions to ODEs Solve the following ODEs.

(a) $x'(t) + \lambda x(t) = 0$, with $x(0) = x_0$.

We can express the ODE as

$$e^{-\lambda t}(x(t)e^{\lambda t})' = 0 \quad \Longleftrightarrow \quad (x(t)e^{\lambda t})' = 0.$$

Integrating, we reach that

$$x(t)e^{\lambda t} = C \quad \Longleftrightarrow \quad x(t) = Ce^{-\lambda t}.$$

Imposing $x(0) = x_0$, we obtain $C = x_0$, and thus

$$x(t) = x_0 e^{-\lambda t}.$$

(b) $x'(t) + \lambda x(t) = 1$, with $x(0) = x_0$.

We can express the ODE as

$$e^{-\lambda t}(x(t)e^{\lambda t})' = 1 \quad \Longleftrightarrow \quad (x(t)e^{\lambda t})' = e^{\lambda t}.$$

Integrating, we reach that

$$x(t)e^{\lambda t} = \frac{1}{\lambda}e^{\lambda t} + C \quad \Longleftrightarrow \quad x(t) = \frac{1}{\lambda} + Ce^{-\lambda t}.$$

Imposing $x(0) = x_0$, we obtain $C = x_0 - \frac{1}{\lambda}$, and thus

$$x(t) = \frac{1}{\lambda} + \left(x_0 - \frac{1}{\lambda}\right) e^{-\lambda t}.$$

(c) $x'(t) + x(t) = t$, with $x(0) = 1$. We can express the ODE as

$$e^{-t}(x(t)e^{t})' = t \quad \Longleftrightarrow \quad (x(t)e^{t})' = te^{t}.$$

Integrating (by parts), we reach that

$$x(t)e^{t} = e^{t}(t-1) + C \quad \Longleftrightarrow \quad x(t) = t - 1 + Ce^{-t}.$$

Imposing $x(0) = 1$, we obtain $C = 2$, and thus

$$x(t) = t - 1 + 2e^{-t}.$$

(d) $x'(t) + x(t) = e^t$, with $x(0) = 1$. We can express the ODE as

$$e^{-t}(x(t)e^t)' = e^t \quad \iff \quad (x(t)e^t)' = e^{2t}.$$

Integrating , we reach that

$$x(t)e^t = \frac{1}{2}e^{2t} + C \quad \iff \quad x(t) = \frac{1}{2}e^t + Ce^{-t}.$$

Imposing $x(0) = 1$, we obtain $C = \frac{1}{2}$, and thus

$$x(t) = \frac{e^t + e^{-t}}{2}.$$

(e) $x''(t) + \lambda^2 x(t) = 0$, find a general solution.

The characteristic polynomial is $p(x) = x^2 + \lambda^2$, with roots $\pm|\lambda|i$. Thus, the general solution is of the form

$$x(t) = C_1 e^{i|\lambda|t} + C_2 e^{-i|\lambda|t},$$

for some constants C_1 and C_2. Alternatively, we can write

$$x(t) = B_1 \sin(\lambda t) + B_2 \cos(\lambda t),$$

for some constants B_1 and B_2.

1.4. Nonexistence of solutions Show that there is not a smooth function $u : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{cases} u_x = xy\,, \\ u_y = x^2\,. \end{cases}$$

Assume that u is a solution of the given system of partial differential equations. Then we have

$$\begin{aligned} u_{yx} &= \partial_y(u_x) = \partial_y(xy) = x\,, \\ u_{xy} &= \partial_x(u_y) = \partial_x(x^2) = 2x\,. \end{aligned}$$

Since the two values are different, we have found a contradiction (as Schwarz's theorem states that for smooth function we can exchange the order of derivation) and this proves that u cannot solve the system of equations.

1.5. Multiple Choice

(a) The correct answer is Linear. In fact it suffices to notice that $\Delta v = \Delta(e^u) = \sum_{i=1}^n (e^u)_{x_i x_i} = \sum_{i=1}^n (e^u u_{x_i})_{x_i} = \sum_{i=1}^n (e^u u_{x_i} u_{x_i} + e^u u_{x_i x_i}) = e^u\Big(\nabla u \cdot \nabla u + \Delta u\Big) = 0$, where the last identity holds because u solves $\nabla u \cdot \nabla u + \Delta u = 0$ by assumption. Therefore, v solves $\Delta v = 0$, which is linear. Alternatively, since $\ln(v) = u$, one can take advantage of the equation for u computing

$$
\begin{aligned}
0 = \nabla u \cdot \nabla u + \Delta u &= \nabla(\ln(v)) \cdot \nabla(\ln(v)) + \Delta(\ln(v)) \\
&= \frac{1}{v^2}\nabla v \cdot \nabla v + \sum_{i=1}^n (\ln(v))_{x_i x_i} = \frac{1}{v^2}\nabla v \cdot \nabla v + \sum_{i=1}^n \Big(\frac{v_{x_i}}{v}\Big)_{x_i} \\
&= \frac{1}{v^2}\nabla v \cdot \nabla v - \frac{1}{v^2}\nabla v \cdot \nabla v + \frac{\Delta v}{v},
\end{aligned}
$$

implying $\frac{\Delta v}{v} = 0$, and hence $\Delta v = 0$.

(b) The correct answer is Quasi linear. The computation goes as follows

$$
\begin{aligned}
u = \operatorname{div}(\nabla(u^2)) = \operatorname{div}(2u\nabla u) = 2\sum_{i=1}^n (u u_{x_i})_{x_i} = 2\sum_{i=1}^n (u_{x_i} u_{x_i} + u u_{x_i x_i}) \qquad (1) \\
= 2(\nabla u \cdot \nabla u + u\Delta u). \qquad (2)
\end{aligned}
$$

As a side remark, notice that the general identity $\operatorname{div}(\nabla v) = \Delta v$ holds for any C^2 function v.