2.1. Method of characteristics Solve the following equations using the method of characteristics.

(a) $u_x + u_y = 1$, with $u(x,0) = f(x)$.

Sol. The characteristic equations and parametric initial conditions are given by

$$\begin{aligned} &x_t(t,s) = 1, \quad y_t(t,s) = 1, \quad u_t(t,s) = 1, \\ &x(0,s) = s, \quad y(0,s) = 0, \quad u(0,s) = f(s). \end{aligned}$$

Solving each of the ODEs separately, we obtain that

$$x(t,s) = t + s, \quad y(t,s) = t, \quad u(t,s) = t + f(s).$$

Thus, inverting the transformation $(x(t,s), y(t,s))$ we have that

$$t = y, \quad s = x - y,$$

so that

$$u = t + f(s) = y + f(x - y).$$

That is, $u(x,y) = y + f(x-y)$ is our solution.

(b) $xu_x + (x+y)u_y = 1$, with $u(1,y) = y^2$.

Sol. The characteristic equations and parametric initial conditions are given by

$$\begin{aligned} &x_t(t,s) = x, \quad y_t(t,s) = x + y, \quad u_t(t,s) = 1, \\ &x(0,s) = 1, \quad y(0,s) = s, \qquad u(0,s) = s^2. \end{aligned}$$

Solving for $x(t,s)$ we obtain $x(t,s) = e^t$. Then, $y_t = e^t + y$, with $y(0,s) = s$. The ODE can be rewritten as $e^t \partial_t(e^{-t}y) = e^t$, or $\partial_t(e^{-t}y) = 1$. Solving for $e^{-t}y$ with initial condition $e^0 y(0,s) = s$, we get $e^{-t}y(t,s) = t + s$, and thus $y(t,s) = (t+s)e^t$.

Finally, solving for $u(t,s)$ we get $u(t,s) = t + s^2$. That is

$$x(t,s) = e^t, \quad y(t,s) = (t+s)e^t, \quad u(t,s) = t + s^2.$$

Inverting the transformation $(x(t,s), y(t,s))$ we have that

$$t = \log(x), \quad s = \frac{y}{x} - \log(x),$$

defined for $x > 0$. Substituting in u,

$$u(x,y) = \log(x) + \left(\frac{y}{x} - \log(x)\right)^2.$$

(c) $u_x - 2xyu_y = 0$, with $u(0,y) = y$.

Sol. The characteristic equations and parametric initial conditions are given by

$$\begin{aligned} &x_t(t,s) = 1, \quad y_t(t,s) = -2xy, \quad u_t(t,s) = 0, \\ &x(0,s) = 0, \quad y(0,s) = s, \qquad\quad u(0,s) = s. \end{aligned}$$

Solving for x we get $x(t,s) = t$. Solving for y then, we obtain $y_t(t,s) = -2ty$, that gives $y(t,s) = se^{-t^2}$. Solving for u, we have $u(t,s) = s$. Inverting the transformation $(x(t,s), y(t,s))$ we have that

$$t = x, \quad s = ye^{t^2},$$

so that

$$u(x,y) = s = ye^{x^2}.$$

(d) $yu_x - xu_y = 0$, with $u(x,0) = g(x^2)$ for all $x > 0$.

Sol. The characteristic equations and parametric initial conditions are given by

$$\begin{aligned} &x_t(t,s) = y, \quad y_t(t,s) = -x, \quad u_t(t,s) = 0, \\ &x(0,s) = s, \quad y(0,s) = 0, \qquad u(0,s) = g(s^2), \text{ where } s > 0. \end{aligned}$$

Differentiating in t one more time the equation for x we obtain that $x_{tt} = y_t = -x$, and hence $x(t,s) = s\cos(t) + A\sin(t)$, where A is a constant possibly depending on s. Integrating now $y_t = -x = -s\cos(t) - A\sin(t)$ in t we have that $y(s,t) = -s\sin(t) + A\cos(t)$. The initial condition for y tells us that $0 = y(0,s) = A$. Hence $x(t,s) = s\cos(t)$ and $y(t,s) = -s\sin(t)$. Finally, since

$$s^2 = x(t,s)^2 + y(t,s)^2,$$

we conclude that $u(x(t,s), y(t,s)) = g(s^2) = g(x^2 + y^2)$. Observe that the graph of u is radial, which means that depends only on the distance from the origin. Also, notice that in general u can have a point of non-differentiability at $(x,y) = (0,0)$ (try to picture what happens in the simple case $g(s^2) := 1 + s$). Put differently, u is a strong solution only in $\mathbb{R}^2 \setminus \{0\}$.

2.2. Find a solution Consider the PDE

$$xu_x + yu_y = -2u.$$

Find a solution to the previous PDE such that $u \equiv 1$ on the unit circle.

Sol. The condition for $u \equiv 1$ on the unit circle can be translated into $u(\cos(s), \sin(s)) = 1$ for $0 \le s < 2\pi$. Thus, the characteristic equations and parametric initial conditions are given by

$$\begin{aligned} x_t(t,s) &= x, & y_t(t,s) &= y, & u_t(t,s) &= -2u, \\ x(0,s) &= \cos(s), & y(0,s) &= \sin(s), & u(0,s) &= 1. \end{aligned}$$

Solving each of the ODEs separately, we obtain that

$$x(t,s) = \cos(s)e^t, \quad y(t,s) = \sin(s)e^t, \quad u(t,s) = e^{-2t}.$$

Now notice that $x^2 + y^2 = e^{2t} = \frac{1}{u}$. Therefore,

$$u(x,y) = \frac{1}{x^2 + y^2}$$

is a solution to the PDE, defined for $(x,y) \neq (0,0)$.

2.3. Multiple choice

(a) The expression $f(u_{xxx}) = u_z + 5$ describes a quasilinear PDE of order 3 if and only if

- ☐ f is linear
- ☐ f is invertible
- ☐ f is constant

Sol. The correct answer is: f is invertible. In fact suppose f is bijective, then one can rewrite the PDE in the equivalent reformulation $u_{xxx} = f^{-1}(u_z + 5)$, which is clearly quasilinear. Conversely, if the PDE is quasilinear, then one can reformulate it as $u_{xxx} = g(u_z)$, for some function g. In particular, we must have that $g = f^{-1}$, and hence f must be bijective.

(b) Consider the PDE $yu_x - x^2 u_y = 0$ coupled with the boundary condition $u(x,y) = 2$ on $\{(x,y) : x^3 + 1 = y\}$. Then, the initial curve $\Gamma(s) = \{x_o(s), y_0(s), \tilde{u}_0(s)\}$ needed to start applying the Method of Characteristic is given by

- ☐ $x_0(s) = s^3 + 1$, $y_0(s) = s$ and $\tilde{u}_0(s) = 2$, $s \in \mathbb{R}$
- ☐ $x_0(s) = s$, $y_0(s) = s^3 + 1$ and $\tilde{u}_0(s) = 2$, $s \in \mathbb{R}$
- ☐ $x_0(s) = s^{1/3}$, $y_0(s) = s + 1$ and $\tilde{u}_0(s) = 2$, $s \in \mathbb{R}$

Sol. The graph $\{(x,y) : x^3 + 1 = y\}$ can be parametrised both as $s \mapsto (s, s^3 + 1)$ and as $s \mapsto (s^{1/3}, s + 1)$ (by substituting $s \rightsquigarrow s^{1/3}$), the correct answers are the second *and* the third. This is an example of not uniqueness of parametrisation.