3.1. Characteristic method and initial conditions Consider the transport equation

$$xu_y - yu_x = 0.$$

For each of the following initial conditions, solve the problem in $y \geq 0$ whenever it is possible. If it is not, explain why.

(a) $u(x,0) = x^2$.

Sol. The characteristic equations and parametric initial conditions are given by

$$\begin{aligned} &x_t(t,s) = -y, \quad && y_t(t,s) = x, \quad && \tilde{u}_t(t,s) = 0, \\ &x(0,s) = s, && y(0,s) = 0, && \tilde{u}(0,s) = s^2. \end{aligned}$$

Notice that, in particular, $x_{tt}(t,s) = -x$ and $y_{tt}(t,s) = -y$. Imposing the initial conditions, we obtain that

$$x(t,s) = s\cos(t), \quad y(t,s) = s\sin(t), \quad \tilde{u}(t,s) = s^2.$$

That is, $\tilde{u}(t,s) = x(t,s)^2 + y(t,s)^2$, or

$$u(x,y) = x^2 + y^2,$$

which is well defined for $y \geq 0$.

(b) $u(x,0) = x$.

Sol. As before, we obtain

$$x(t,s) = s\cos(t), \quad y(t,s) = s\sin(t), \quad \tilde{u}(t,s) = s.$$

(Notice that, if we want to work in $y \geq 0$, we have to impose that $0 \leq t \leq \pi$ for $s \geq 0$ and $-\pi \leq t \leq 0$ for $s \leq 0$.) This yields $\tilde{u}(t,s)^2 = x(t,s)^2 + y(t,s)^2$. Nonetheless, heuristically, we cannot recover uniquely $\tilde{u}$ from this expression, since for some initial values $\tilde{u}$ is negative, and for others is positive.

Let us see that, in fact, the equation is not solvable. Notice that, for any $s > 0$, the characteristic curves $(x(t,s), y(t,s))$ will intersect the initial curve $\{(x,0) : x \in \mathbb{R}\}$ at two points, $(s,0)$ and $(-s,0)$, for $t = 0$ and $t = \pi$ respectively. That is,

$$u(s,0) = \tilde{u}(0,s) = \tilde{u}(\pi,s) = u(-s,0).$$

But $u(s,0) = s$ and $u(-s,0) = -s$ from the initial conditions. Contradiction. The equation is not solvable.

(c) $u(x,0) = x$ for $x > 0$.

Sol. Notice that, from before, now $\tilde{u}(t,s)^2 = x(t,s)^2 + y(t,s)^2$ inverting as $\tilde{u}(t,s) = \sqrt{x(t,s)^2 + y(t,s)^2}$, then

$$u(x,y) = \sqrt{x^2 + y^2}$$

fulfils the initial condition $u(x,0) = x$ for $x > 0$, so that the equation is solvable.

Heuristically, notice that, since the initial condition is only crossed once (the other crossing point from the previous exercise was for $x < 0$), now the equation is solvable.

3.2. Characteristic method and transversality condition Consider the transport equation

$$yu_x + uu_y = x.$$

(a) Solve the problem with initial condition $u(s,s) = -2s$, for $s \in \mathbb{R}$. For what domain of s does the transversality condition hold?

Sol. The characteristic equations and parametric initial conditions are given by

$$\begin{array}{lll} x_t(t,s) = y(t,s), & y_t(t,s) = \tilde{u}(t,s), & \tilde{u}_t(t,s) = x(t,s), \\ x(0,s) = s, & y(0,s) = s, & u(0,s) = -2s. \end{array}$$

Notice that, if we define $w(t,s) := x(t,s) + y(t,s) + \tilde{u}(t,s)$, then $w_t(t,s) = w(t,s)$ and $w(0,s) = 0$. That is, $w(t,s) \equiv 0$ for all s, and therefore,

$$u(x,y) = -x - y.$$

Regarding the transversality condition, let us check:

$$J = \begin{vmatrix} x_t(0,s) & y_t(0,s) \\ x_s(0,s) & y_s(0,s) \end{vmatrix} = \begin{vmatrix} y(0,s) & \tilde{u}(0,s) \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} s & -2s \\ 1 & 1 \end{vmatrix} = 3s \neq 0, \quad \text{if} \quad s \neq 0.$$

That is, the transversality condition holds if $s \neq 0$.

(b) Check the transversality condition with the initial value $u(s,s) = s$. What is occurring in this case?

Sol. The characteristic equations and parametric initial conditions are given by

$$\begin{array}{lll} x_t(t,s) = y(t,s), & y_t(t,s) = \tilde{u}(t,s), & \tilde{u}_t(t,s) = x(t,s), \\ x(0,s) = s, & y(0,s) = s, & u(0,s) = s. \end{array}$$

Regarding the transversality condition, let us check:

$$J = \begin{vmatrix} x_t(0,s) & y_t(0,s) \\ x_s(0,s) & y_s(0,s) \end{vmatrix} = \begin{vmatrix} y(0,s) & \tilde{u}(0,s) \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} s & s \\ 1 & 1 \end{vmatrix} = 0.$$

The transversality condition never holds. What is occurring is that the solution to the characteristic equations is (se^t, se^t, se^t), which coincides with the initial curve. In other words, from the PDE and the initial condition, we get no information on u outside of the line $s \mapsto (s, s, s)$.

Therefore, the problem is under-determined, and it has infinitely many solutions.

(c) Define

$$w_1 := x + y + u, \quad w_2 := x^2 + y^2 + u^2, \quad w_3 = xy + xu + yu.$$

Show that $w_1(w_2 - w_3)$ is constant along the characteristic curves.

Sol. characteristic curves fulfill the equations

$$x_t(t) = y(t), \quad y_t(t) = \tilde{u}(t), \quad \tilde{u}_t(t) = x(t)$$

(we removed the parameter s, since we will not care about initial value conditions for this part).

In particular, if we consider w_i along the curves, we can take $\tilde{w}_i(t) := w_i(x(t), y(t), \tilde{u}(t))$. We want to show that $\frac{d}{dt}\tilde{w}_1(\tilde{w}_2 - \tilde{w}_3) = 0$. Indeed:

$$\begin{aligned}
\frac{d\tilde{w}_1(t)}{dt} &= \tilde{w}_1(t), \\
\frac{d\tilde{w}_2(t)}{dt} &= 2x(t)u(t) + 2y(t)\tilde{u}(t) + 2x(t)y(t) = 2\tilde{w}_3(t), \\
\frac{d\tilde{w}_3(t)}{dt} &= y^2(t) + x(t)\tilde{u}(t) + x^2(t) + y(t)\tilde{u}(t) + \tilde{u}^2(t) + y(t)x(t) \\
&= \tilde{w}_2(t) + \tilde{w}_3(t).
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{d}{dt}\tilde{w}_1(\tilde{w}_2 - \tilde{w}_3) &= \left(\frac{d}{dt}\tilde{w}_1\right)(\tilde{w}_2 - \tilde{w}_3) + \tilde{w}_1\left(\frac{d}{dt}\tilde{w}_2 - \frac{d}{dt}\tilde{w}_3\right) \\
&= \tilde{w}_1(\tilde{w}_2 - \tilde{w}_3) + \tilde{w}_1(2\tilde{w}_3 - \tilde{w}_2 - \tilde{w}_3) = 0,
\end{aligned}$$

as we wanted to see.