5.1. Finding shock waves

Consider the transport equation

$$u_y + u^2 u_x = 0,$$

with initial condition $u(x,0) = 1$ for $x \leq 0$, $u(x,0) = 0$ for $x \geq 1$, and

$$u(x,0) = \sqrt{1-x} \quad \text{for} \quad 0 < x < 1.$$

(a) Find the solution using the method of characteristics. Up to which time is the solution defined in a classical sense?

Sol. First notice that the solution has a derivative blowing up for all times $y \geq 0$, since the initial datum is non-smooth. However, we can solve the initial value problem up until the characteristics intersect.

The characteristic equations and parametric initial conditions are given by

$$\begin{aligned} &x_t(t,s) = \tilde{u}^2, \quad &&y_t(t,s) = 1, \quad &&\tilde{u}_t(t,s) = 0, \\ &x(0,s) = s, &&y(0,s) = 0, &&\tilde{u}(0,s) = h(s), \end{aligned}$$

where $h(s) = 1$ if $s \leq 0$, $h(s) = 0$ if $s \geq 1$, and $h(s) = \sqrt{1-s}$ otherwise. Notice that $y(t,s) = t$, $\tilde{u}(t,s) = h(s)$, and $x(t,s) = s + h(s)^2 t$.

Let us invert the characteristics:

- If $s \leq 0$, $x(t,s) = s + y(t,s)$, $(t,s) = (y, x-y)$. Note that, since $s \leq 0$, we have $x \leq y$.
- If $0 < s < 1$, $x(t,s) = s + (1-s)t$, and $(t,s) = \left(y, \frac{x-y}{1-y}\right)$. Since $0 < s < 1$, then $0 < \frac{x-y}{1-y} < 1$ implies either $y < x < 1$, or $y > x > 1$.
- If $s \geq 1$, $(t,s) = (y,x)$. In this case, $x \geq 1$.

The characteristics are intersecting at the point $(x,y) = (1,1)$. Thus, the solution is defined up to $y = 1$. For $y \leq 1$, it is:

$$u(x,y) = \begin{cases} 1 & \text{if} \quad x \leq y \\ 0 & \text{if} \quad x \geq 1 \\ h\left(\frac{x-y}{1-y}\right) = \sqrt{\frac{1-x}{1-y}} & \text{if} \quad y < x < 1 \end{cases}$$

(b) Find a weak solution for all times $y \geq 0$.

Sol. Let us express the equation in the form

$$u_y + \partial_x \left(F(u)\right) = 0.$$

In this case, a simple inspection yields $F(u) = \frac{1}{3}u^3$.

We are trying to solve $u_y + \partial_x (F(u)) = 0$ with initial value $u(x,1) = 1$ if $x < 1$, and $u(x,1) = 0$ if $x > 1$.

We just have to compute the slope of the curve of discontinuity, given by

$$\gamma_y(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-} = \frac{1}{3}\frac{0 - 1^3}{0 - 1} = \frac{1}{3}.$$

Thus, the solution for $y \geq 1$ is given by

$$u(x,y) = \begin{cases} 1 & \text{if } x < \frac{1}{3}(y-1) + 1 \\ 0 & \text{if } x > \frac{1}{3}(y-1) + 1, \end{cases}$$

and for $y \leq 1$, as before.

5.2. Weak solutions

Consider the transport equation

$$u_y + \frac{1}{2}\partial_x \left(u^2\right) = 0. \tag{1}$$

(a) Suppose that u is a classical solution to the previous transport equation. What equation does u^2 fulfil? Write it in the form

$$v_y + \partial_x (F(v)) = 0, \tag{2}$$

for some appropriate F.

Sol. Let $v = u^2$. We first compute

$$v_y = \partial_y(u^2) = 2uu_y, \qquad v_x = 2uu_x.$$

This gives, using the equation for u, namely $u_y + uu_x = 0$,

$$v_y = 2uu_y = -2u^2 u_x = -uv_x = -v^{\frac{1}{2}} v_x,$$

and the last term can be rewritten as $\partial_x(F(v)) = F'(v)v_x$ if $F'(v) = v^{\frac{1}{2}}$, hence $F(v) = \frac{2}{3}v^{\frac{3}{2}}$.

(b)

(c) Consider the weak solution of Equation (1) given by

$$w(x,y) = \begin{cases} 3 & \text{if } x < \frac{3}{2}y - 1 \\ 0 & \text{if } x > \frac{3}{2}y - 1. \end{cases}$$

Show that w^2 is not a weak solution of (2). Can you explain what is the problem?

Sol. Let us denote

$$w(x,y) = \begin{cases} 3 & \text{if } x < \frac{3}{2}y - 1 \\ 0 & \text{if } x > \frac{3}{2}y - 1. \end{cases}$$

We have to check whether w^2 is a weak solution to

$$v_y + \frac{2}{3}\partial_x\left(v^{\frac{3}{2}}\right) = 0,$$

with $v(x,0) = 9$ for $x < -1$, and $v(x,0) = 0$ for $x > -1$. Notice that the curve of discontinuity of weak solutions to our Cauchy problem is given by is $x = \gamma(y)$ where

$$\gamma_y(y) = \frac{F(v^+) - F(v^-)}{v^+ - v^-} = \frac{2}{3}\frac{0 - 9^{\frac{3}{2}}}{0 - 9} = 2.$$

On the other hand, the curve of discontinuity of w^2 is of slope $\frac{3}{2}$. That is, w^2 cannot be a weak solution to (2).

What is happening is that to derive (2) from the equation $u_y + uu_x = 0$ we have applied the chain rule (for instance to say that $\partial_x(u^2) = 2uu_x$) which is rigorous only for smooth functions. This exercise actually shows that the chain rule is false for functions with jumps and one needs to be careful when dealing with weak solutions.

5.3. Weak solutions II

Consider the equation

$$e^{-u}u_x + u_y = 0,$$

with initial value $u(x,0) = 0$ if $x < 0$, and $u(x,0) = \alpha > 0$ if $x > 0$.

(a) Find a weak solution for any $\alpha > 0$ with a single discontinuity for $y \geq 0$.

Sol. Let us express our equation in the form (2). By inspection, we reach that $F(t) = -e^{-t}$. The curve of discontinuity of our Cauchy problem is then $x = \gamma(y)$ with

$$\gamma_y(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-} = \frac{-e^{-\alpha} + 1}{\alpha} > 0,$$

and our solution is given by

$$u(x,y) = \begin{cases} 0 & \text{if } x < \frac{1-e^{-\alpha}}{\alpha}y \\ \alpha & \text{if } x > \frac{1-e^{-\alpha}}{\alpha}y. \end{cases}$$

(b) Show that such solution fulfils the entropy condition for all $\alpha > 0$.

Sol. The entropy condition is

$$F_u(u_-) > \gamma_y > F_u(u^+), \qquad \text{where} \quad F_u(t) = e^{-t}.$$

That is,

$$1 > \frac{1-e^{-\alpha}}{\alpha} > e^{-\alpha} \quad \Longleftrightarrow \quad \alpha > 1 - e^{-\alpha} > \alpha e^{-\alpha}$$

The first inequality holds, since $\frac{d}{d\alpha}\alpha = 1 > e^{-\alpha} = \frac{d}{d\alpha}(1-e^{-\alpha})$ for $\alpha > 0$, and for $\alpha = 0$ they coincide.

The second inequality corresponds to checking

$$f(\alpha) := 1 - e^{-\alpha} > \alpha e^{-\alpha} =: g(\alpha),$$

for $\alpha > 0$. Both sides coincide for $\alpha = 0$, $f(0) = g(0)$. Then, it is enough to check that $f'(\alpha) > g'(\alpha)$ for all $\alpha > 0$. Indeed

$$f'(\alpha) = e^{-\alpha} > e^{-\alpha} - \alpha e^{-\alpha} = g'(\alpha),$$

and we are done.

5.4. Multiple Choice

(a) The second order linear PDE given by

$$u_x + x^2 u_{xx} + 2x\sin(y)u_{xy} - \cos^2(y)u_{yy} + e^x = 0,$$

is

- ☐ Everywhere hyperbolic
- ☐ Parabolic for $\{y : \cos(y) = 0 \text{ or } \sin(y) = 0\} = \{k\frac{\pi}{2} :\in \mathbb{Z}\}$ and hyperbolic elsewhere
- ☐ Parabolic in $x = 0$, and hyperbolic elsewhere

Sol. The coefficients associated to this PDE are $a = x^2$, $b = x\sin(y)$ and $c = -\cos^2(y)$. The discriminant is given by $\delta = b^2 - ac = x^2\sin^2(y) + x^2\cos^2(y) = x^2(\sin^2(y) + \cos^2(y)) = x^2$. Hence, the PDE is parabolic when $\delta = x^2 = 0$, and hyperbolic otherwise because $x^2 > 0$ if $x \neq 0$. The correct answer is the third one.

(b) The following conservation law

$$\begin{cases} u_y + f(u)_x = 0, \\ u(x,0) = c > 0 \text{ for } \{x < 0\} \text{ and } u(x,0) = 0 \text{ for } \{x \geq 0\} \end{cases}$$

has a shock curve of slope equal to 8 if

☐ $c = 2$ and $f(u) = u^4$

☐ $c = 2$ and $f(u) = -u^4$

☐ $c = 1$ and $f(u) = 2u^2 + 6u - 1$

Sol. We need to verify the Rankine-Hugoniot condition, which in this case reads

$$\gamma' = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = -\frac{f(0) - f(c)}{0 - c} = \begin{cases} \frac{-2^4}{-2} = 8, & \text{when } c = 2, f(u) = u^3, \\ \frac{2^4}{-2} = -8, & \text{when } c = 2, f(u) = -u^3, \\ \frac{-1-(2+6-1)}{-1} = 8, & \text{when } c = 1, f(u) = 2u^2 + 6u - 1. \end{cases}$$

The correct answers are the first and third.