### 6.1. Wave equation

Let $c > 0$, and consider the wave equation posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= \sin(x), & x \in \mathbb{R}, \\ u_t(x,0) &= 0, & x \in \mathbb{R}. \end{cases}$$

**(a)** Solve the Cauchy problem. Identify the forward and the backward wave, and express the solution with separated variables, that is, $u(x,t) = v(x)w(t)$ for some functions $v$ and $w$. (*Hint: Recall that* $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$.)

**Sol.** We use d'Alembert's formula. That is,

$$u(x,t) = \frac{\sin(x+ct) + \sin(x-ct)}{2}.$$

The forward wave is then $\frac{\sin(x-ct)}{2}$ and the backward wave is $\frac{\sin(x+ct)}{2}$.

Now, from the hint we deduce that $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$, so that our solution is

$$u(x,t) = \sin(x)\cos(ct).$$

**(b)** Show that $u$ is $\frac{2\pi}{c}$-periodic in the $t$ variable. That is, show that $u(x,t) = u(x,t+T)$ where $T = \frac{2\pi}{c}$.

**Sol.** This can be immediately seen from any of the both expressions of $u$, since both sin and cos are $2\pi$-periodic. For example, using that cos is $2\pi$-periodic,

$$u(x,t+T) = \sin(x)\cos(ct+2\pi) = \sin(x)\cos(ct) = u(x,t),$$

where we recall that $T = \frac{2\pi}{c}$.

### 6.2. Odd initial data

Consider the general wave equation posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= f(x), & x \in \mathbb{R}, \\ u_t(x,0) &= g(x), & x \in \mathbb{R}. \end{cases}$$

Suppose that both $f$ and $g$ are odd functions (that is, $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}$). Show that the solution $u$ is also an odd function in $x$, for each time $t > 0$ (that is, $u(-x,t) = -u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$).

**Sol.** By d'Alembert's formula

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

That is,

$$u(-x,t) = \frac{f(-x+ct)+f(-x-ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} g(s)\,ds.$$

Let us study each part separately. On the one hand, notice that, since $f$ is odd, $f(-x+ct) = -f(x-ct)$ and $f(-x-ct) = -f(x+ct)$, so

$$\frac{f(-x+ct)+f(-x-ct)}{2} = \frac{-f(x-ct)-f(x+ct)}{2} = -\frac{f(x+ct)+f(x-ct)}{2}.$$

On the other hand, we deal with the integral term. We change variables, $s \mapsto -\xi$, so that $ds \mapsto -d\xi$ and if $s$ goes from $-x-ct$ to $-x+ct$ then $\xi$ goes from $x+ct$ to $x-ct$. That is,

$$\int_{-x-ct}^{-x+ct} g(s)\,ds = -\int_{x+ct}^{x-ct} g(-\xi)\,d\xi = \int_{x-ct}^{x+ct} g(-\xi)\,d\xi = -\int_{x-ct}^{x+ct} g(\xi)\,d\xi,$$

where in the last equality we are using that $g$ is odd. Putting everything together,

$$\begin{aligned} u(-x,t) &= \frac{f(-x+ct)+f(-x-ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} g(s)\,ds \\ &= -\frac{f(x+ct)+f(x-ct)}{2} - \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi \\ &= -u(x,t), \end{aligned}$$

as we wanted to see.

### 6.3. Zero boundary condition

Use the previous exercise to solve the following Cauchy problem posed for $x > 0$ and $t > 0$, with zero boundary condition at $x = 0$,

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x,t) \in (0,\infty)\times(0,\infty), \\ u(0,t) = 0, & t \in (0,\infty), \\ u(x,0) = x^2, & x \in (0,\infty), \\ u_t(x,0) = 0, & x \in (0,\infty). \end{cases}$$

**Sol.** Notice that our problem is now posed on the half-line, $x > 0$, with zero boundary condition at $x = 0$ for all times $t > 0$. The initial value is $x^2$, which when evaluated at 0 is consistent with the boundary condition.

The previous exercise tells us that solutions starting from an odd initial value, remain odd at all times. In particular, we know that continuous odd functions must be 0 at 0: that is, if $f(-x) = -f(x)$ for all $x$, then for $x = 0$ we get $f(0) = -f(0)$ which means $f(0) = 0$. Then, the zero boundary condition at all times posed at $x = 0$ will hold if the solution is odd at all times. Thus, thanks to the previous exercise, it will be enough to solve the problem in the whole line

$$\begin{cases} u_{tt} - u_{xx} &= 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= f(x), & x \in \mathbb{R}, \\ u_t(x,0) &= 0, & x \in \mathbb{R}, \end{cases}$$

where $f(x) = x^2$ if $x > 0$ and $f(x) = -x^2$ if $x \leq 0$, is the odd extension of $x^2$ to the whole $\mathbb{R}$. Alternatively, we can write $f(x) = x|x|$. Thus, our solution, by d'Alembert formula (which works only if the domain is the whole real line $\mathbb{R}$), is given by

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} = \frac{(x+t)|x+t| + (x-t)|x-t|}{2}.$$

More explicitly, we can separate in three different cases:

- If $x \geq t$, then $x + t \geq 0$ and $x - t \geq 0$, so that
$$u(x,t) = \frac{(x+t)^2 + (x-t)^2}{2} = x^2 + t^2.$$
- If $-t < x < t$, then $x - t < 0$ and $x + t > 0$, so that
$$u(x,t) = \frac{(x+t)^2 - (x-t)^2}{2} = 2xt.$$
- If $x \leq -t$, then $x + t \leq 0$ and $x - t \leq 0$, so that
$$u(x,t) = -\frac{(x+t)^2 + (x-t)^2}{2} = -x^2 - t^2.$$

By construction, it is clear that the function $u$ restricted to $\{x \geq 0\}$ solves the PDE on the half line as wished.

### 6.4. Time reversible

Consider the Cauchy problem posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= f(x), & x \in \mathbb{R}, \\ u_t(x,0) &= g(x), & x \in \mathbb{R}. \end{cases}$$

Let $\tilde{u}(x,t) := u(x,-t)$. Show that $\tilde{u}(x,t)$ solves the Cauchy problem posed for $-\infty < x < \infty$ and $t < 0$,

$$\begin{cases} \tilde{u}_{tt} - c^2\tilde{u}_{xx} &= 0, & (x,t) \in \mathbb{R} \times (-\infty, 0), \\ \tilde{u}(x,0) &= f(x), & x \in \mathbb{R}, \\ \tilde{u}_t(x,0) &= -g(x), & x \in \mathbb{R}. \end{cases}$$

That is, we are showing that the wave equation is reversible in time. If a function solves a wave equation, the same function with time reversed also solves a the wave equation with the same initial condition and opposite initial velocity.

**Sol.** We just need to check the properties one by one.

First notice that

$$\tilde{u}(x,0) = u(x,0) = f(x)$$

and

$$\tilde{u}_t(x,0) = \frac{d}{dt}(u(x,-t))\Big|_{t=0} = -u_t(x,0) = -g(x),$$

so that the initial conditions hold. Now, since $u$ is defined for $(x,t) \in \mathbb{R} \times (0,\infty)$, then $\tilde{u}$ is defined for $(x,t) \in \mathbb{R} \times (-\infty, 0)$. Finally, notice that

$$\tilde{u}_{tt}(x,t) = (u(x,-t))_{tt} = -(u_t(x,-t))_t = u_{tt}(x,-t)$$

and similarly

$$\tilde{u}_{xx}(x,t) = (u(x,-t))_{xx} = u_{xx}(x,-t).$$

Therefore,

$$\tilde{u}_{tt}(x,t) - c^2\tilde{u}_{xx}(x,t) = u_{tt}(x,-t) - c^2 u_{xx}(x,-t) = 0$$

where we are using the original equation, $u_{tt} - c^2 u_{xx} = 0$.

**6.5. Multiple Choice** Determine the correct answer.

**(a)** Consider the one dimensional wave equation given by

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = \arctan(x), & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Then, the asymptotic value of the solution at any $\bar{x} \in \mathbb{R}$ (i.e. $\lim_{t\to+\infty} u(\bar{x},t)$) is equal to

☐ 0

☐ $\pi/2$

- ☐ $\pi/2c$

**Sol.** It suffices to take the limit as $t \to +\infty$ in d'Alembert's formula for a fixed point in space $\bar{x} \in \mathbb{R}$:

$$\lim_{t\to+\infty} \frac{\arctan(\bar{x}+ct) + \arctan(\bar{x}-ct)}{2} + \frac{1}{2c}\int_{\bar{x}-ct}^{\bar{x}+ct} 0\, dy = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = 0.$$

The correct answer is the first one.

**(b)** Given

$$\begin{cases} u_{tt} - \pi^2 u_{xx} = 0, \\ u(x,0) = x^2, & x \in \mathbb{R}, \\ u_t(x,0) = -\sin(x), & x \in \mathbb{R}. \end{cases}$$

the value of $u$ at the point $(x,t) = (\pi, 2)$ is equal to

- ☐ 0
- ☐ $5\pi^2$
- ☐ $3\pi^2$

**Sol.** Once again, we apply d'Alembert's formula

$$\begin{aligned} u(\pi,2) &= \frac{(\pi+2\pi)^2 - (\pi-2\pi)^2}{2} - \frac{1}{2\pi}\int_{\pi-2\pi}^{\pi+2\pi} \sin(y)\, dy \\ &= \frac{2\pi^2 + 8\pi^2}{2} + \frac{1}{2\pi}(\cos(3\pi) - \cos(-\pi)) = 5\pi^2. \end{aligned}$$

The correct answer is the second one.