8.1. Separation of variables

Solve the following equations using the method of separation of variables and superposition principle. To do so, write first a general solution solving the problem with boundary conditions, and then impose the initial values.

(a)

$$\begin{cases} u_t - u_{xx} = 0, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) = 0, & t \in (0,\infty), \\ u(\pi,t) = 0, & t \in (0,\infty), \\ u(x,0) = \sin(2x) + \frac{1}{2}\sin(3x) + 5\sin(5x), & x \in [0,\pi]. \end{cases}$$

Sol. Let us use the method of separation of variables. Let us assume that there exists a solution of the form

$$u(x,t) = X(x)T(t).$$

Then, since $u_t = u_{xx}$, $XT' = X''T$ or

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

Since the left-hand side depends only on x, and the right-hand depends only on t, both expressions must be constant, say, equal to $-\lambda$ for some $\lambda \in \mathbb{R}$. Thus, we have

$$\frac{d^2X}{dx^2} = -\lambda X, \quad x \in (0,\pi)$$
$$\frac{dT}{dt} = -\lambda T, \quad t \in (0,\infty).$$

From the Dirichlet boundary conditions, we have that

$$X(0)T(t) = X(\pi)T(t) = 0, \quad t \in (0,\infty),$$

which, since u is nontrivial, implies that

$$X(0) = X(\pi) = 0.$$

Summarising, the function X solves the eigenvalue problem

$$\frac{d^2X}{dx^2} = -\lambda X, \quad x \in (0,\pi)$$
$$X(0) = X(\pi) = 0.$$

General solutions to the ODE are of the form

- If $\lambda > 0$,
$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$
- If $\lambda = 0$,
$$X(x) = A + Bx.$$
- If $\lambda < 0$,
$$X(x) = A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x).$$

For some constants $A, B \in \mathbb{R}$ to be determined. Imposing the boundary conditions, $X(0) = X(\pi) = 0$, directly yields $A = B = 0$ in the case $\lambda = 0$ (and we have found a trivial solution). Similarly, if $\lambda < 0$, imposing $X(0) = 0$ yields $A = 0$, and imposing $X(\pi) = 0$ yields
$$B\sinh(\sqrt{-\lambda}\pi) = 0,$$
which implies $B = 0$ if $\lambda < 0$, since sinh only vanishes at 0. Thus, if $\lambda < 0$, we have also reached a trivial solution, and $\lambda \leq 0$ is not a possible eigenvalue of our problem.

Let now $\lambda > 0$. First notice that from $X(0) = 0$ we deduce $A = 0$. Then imposing $X(\pi) = 0$,
$$B\sin(\sqrt{\lambda}\pi) = 0.$$
We want a non-trivial solution, so that $B \neq 0$, and $\sin(\sqrt{\lambda}\pi) = 0$, that is, $\sqrt{\lambda} \in \mathbb{N}$ (negative values yield the same eigenvalues and eigenfunctions, so we take them positive). In all, we reach that
$$\lambda = n^2, \quad \text{for} \quad n \in \mathbb{N},$$
that is, $\lambda \in \{1, 4, 9, 16, \dots\}$. That is, taking the value n, we have that $X_n(x) = B_n \sin(nx)$. Solving for T_n such that
$$T_n' = -n^2 T_n, \quad t \in (0, \infty),$$
we reach that
$$T_n(t) = C_n e^{-n^2 t}.$$
That is, for each $n \in \mathbb{N}$, we have found a solution such that has separated variables, and is of the form
$$u_n(x,t) = D_n \sin(nx) e^{-n^2 t}$$
for some constants D_n. By the superposition principle, our solution will be
$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin(nx) e^{-n^2 t}$$

and this is the general solution to the PDE with boundary conditions. Now notice that

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin(nx) = \sin(2x) + \frac{1}{2}\sin(3x) + 5\sin(5x),$$

so that we can identify terms to reach that $D_2 = 1$, $D_3 = \frac{1}{2}$, and $D_5 = 5$ (and all the other coefficients D_n vanish). Our solution is then

$$u(x,t) = \sin(2x)e^{-4t} + \frac{1}{2}\sin(3x)e^{-9t} + 5\sin(5x)e^{-25t}.$$

(b)

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) = 0, & t \in (0,\infty), \\ u(\pi,t) = 0, & t \in (0,\infty), \\ u(x,0) = \sin^3(x), & x \in [0,\pi], \\ u_t(x,0) = \sin(2x), & x \in [0,\pi]. \end{cases}$$

Hint: recall that $4\sin^3(x) = 3\sin(x) - \sin(3x)$.

Sol. We use the method of separation of variables. Let us assume that there exists a solution of the form

$$u(x,t) = X(x)T(t).$$

Then, since $u_{tt} = u_{xx}$, $XT'' = X''T$ or

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}.$$

Since the left-hand side depends only on x, and the right-hand depends only on t, both expressions must be constant, say, equal to $-\lambda$ for some $\lambda \in \mathbb{R}$. Thus, we have

$$\frac{d^2X}{dx^2} = -\lambda X, \quad x \in (0,\pi)$$
$$\frac{d^2T}{dt^2} = -\lambda T, \quad t \in (0,\infty).$$

As before, from the Dirichlet boundary conditions, we have that

$$X(0) = X(\pi) = 0,$$

and arguing as in the previous exercise we reach that $\lambda = n^2$ for $n \in \mathbb{N}$ and $X_n(x) = B_n \sin(nx)$. Thus, we are looking for a particular solution of the form

$$u_n(x,t) = B_n T_n(t) \sin(nx).$$

Notice that now T_n solves the second order ODE $T_n'' = -n^2 T_n$, and therefore

$$T_n(t) = \alpha_n \cos(nt) + \beta_n \sin(nt)$$

for some constants $\alpha_n, \beta_n \in \mathbb{R}$ to be determined. Thus, our general solution, after applying the superposition principle, is going to be

$$u(x,t) = \sum_{n=0}^{\infty} \sin(nx)\left(\gamma_n \cos(nt) + \delta_n \sin(nt)\right),$$

where $\gamma_n = B_n\alpha_n, \delta_n = B_n\beta_n$ are constants yet to be determined.

In order to find the values of γ_n and δ_n we need to impose the boundary conditions. First, let us express $\sin^3(x)$ in terms of the eigenfunctions for x, that is, using the hint:

$$\sin^3(x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x).$$

Thus,

$$u(x,0) = \sum_{n=0}^{\infty} \gamma_n \sin(nx) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x).$$

Also note that

$$u_t(x,t) = \sum_{n=0}^{\infty} \sin(nx)\left(-n\gamma_n \sin(nt) + n\delta_n \cos(nt)\right),$$

so

$$u_t(x,0) = \sum_{n=0}^{\infty} n\delta_n \sin(nx) = \sin(2x).$$

Identifying coefficients we obtain first that $\gamma_1 = \frac{3}{4}$ and $\gamma_3 = -\frac{1}{4}$ (the rest of $\gamma_m = 0$, for $m \notin \{1,3\}$), and $2\delta_2 = 1$, so that $\delta_2 = \frac{1}{2}$ (and the others, again, are 0).

Thus, our solution is

$$u(x,t) = \frac{3}{4}\sin(x)\cos(t) + \frac{1}{2}\sin(2x)\sin(2t) - \frac{1}{4}\sin(3x)\cos(3t).$$

(c)

$$\begin{cases} u_t - u_{xx} = 0, & (x,t) \in (0,\pi)\times(0,\infty), \\ u_x(0,t) = 0, & t \in (0,\infty), \\ u_x(\pi,t) = 0, & t \in (0,\infty), \\ u(x,0) = 1+\cos(x) & x \in [0,\pi]. \end{cases}$$

Sol. We now have a heat equation with Neumann boundary conditions (that is, the condition is imposed on the x-derivative at the boundary, rather than its values). We proceed analogously to the first exercise.

We know that, for some $\lambda \in \mathbb{R}$,

$$\begin{aligned} \frac{d^2X}{dx^2} &= -\lambda X, \quad x \in (0,\pi) \\ \frac{dT}{dt} &= -\lambda T, \quad t \in (0,\infty). \end{aligned}$$

From the Neumann boundary conditions, we have that

$$X'(0)T(t) = X'(\pi)T(t) = 0, \quad t \in (0,\infty),$$

which, since u is nontrivial, implies that

$$X'(0) = X'(\pi) = 0.$$

That is, X solves the eigenvalue problem

$$\begin{aligned} \frac{d^2X}{dx^2} &= -\lambda X, \quad x \in (0,\pi) \\ X'(0) &= X'(\pi) = 0. \end{aligned}$$

General solutions to the ODE are of the form

- If $\lambda > 0$,
$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$
- If $\lambda = 0$,
$$X(x) = A + Bx.$$
- If $\lambda < 0$,
$$X(x) = A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x).$$

For some constants $A, B \in \mathbb{R}$ to be determined. Imposing the boundary conditions, $X'(0) = X'(\pi) = 0$, directly yields $B = 0$ in the case $\lambda = 0$, so that $X(x) = A$ constant is a possible solution.

If $\lambda < 0$, imposing $X'(0) = 0$ yields $B = 0$, and imposing $X'(\pi) = 0$ yields

$$A\sqrt{-\lambda}\sinh(\sqrt{-\lambda}\pi) = 0,$$

which implies $A = 0$ if $\lambda < 0$, since sinh only vanishes at 0. Thus, if $\lambda < 0$, we have reached a trivial solution, and $\lambda < 0$ is not a possible eigenvalue of our problem.

Let now $\lambda > 0$. First notice that from $X'(0) = 0$ we deduce $B = 0$. Then imposing $X'(\pi) = 0$,

$$A\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) = 0.$$

We want a non-trivial solution, so that $A \neq 0$, and $\sin(\sqrt{\lambda}\pi) = 0$, that is,

$$\lambda = n^2, \quad \text{for} \quad n \in \mathbb{N},$$

and $X_n(x) = A_n \cos(nx)$ for $n \in \{0, 1, 2, \dots\}$ (notice that we have added the value $n = 0$ to include the case $\lambda = 0$, which is constant). Solving for T_n such that

$$T_n' = -n^2 T_n, \quad t \in (0, \infty),$$

we reach that

$$T_n(t) = C_n e^{-n^2 t}.$$

That is, by the superposition principle

$$u(x,t) = \sum_{n=0}^{\infty} D_n \cos(nx) e^{-n^2 t}.$$

Imposing the initial value,

$$u(x,0) = \sum_{n=0}^{\infty} D_n \cos(nx) = 1 + \cos(x).$$

Therefore, $D_0 = D_1 = 1$, and $D_n = 0$ for $n \geq 2$. We conclude that

$$u(x,t) = 1 + \cos(x) e^{-t}.$$

8.2. Multiple Choice Determine the correct answer.

(a) Consider the periodic homogeneous wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & (x,t) \in [0,1] \times [0, +\infty) \\ u_x(0,t) = u_x(1,t) = 0, & t > 0, \\ u(x,0) = 1 + 2021\cos(2\pi x), & x \in [0,1], \\ u_t(x,0) = \cos(40\pi x), & x \in [0,1]. \end{cases}$$

Then, for a fixed point $\bar{x} \in [0,1]$, the function $t \mapsto u(\bar{x}, t)$ has period

- ☐ $1/2$
- ☐ $1/40$
- ☐ 2π

(recall that a function f has period $T > 0$ if $f(t+T) = f(t)$ for every t in its domain of definition).

Sol. Applying the formula obtained by separation of variables, the solution u has the form

$$u(x,t) := \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{+\infty} \cos(n\pi x)\Big(A_n \cos(2n\pi t) + B_n \sin(2n\pi t)\Big)$$

where the coefficients A_n and B_n are such that

$$1 + 2021\cos(2\pi x) = \frac{A_0}{2} + \sum_{n=1}^{+\infty} A_n \cos(n\pi x),$$

and

$$\cos(40\pi x) = \frac{B_0}{2} + \sum_{n=1}^{+\infty} 2\pi n B_n \cos(n\pi x).$$

Therefore, $A_n \neq 0$ only when $n = 0, 2$ and $B_n \neq 0$ only when $n = 40$, obtaining

$$u(x,t) = \frac{A_0}{2} + A_2 \cos(2\pi x)\cos(4\pi t) + B_{40}\cos(40\pi x)\sin(80\pi t).$$

Now, since

$$\cos(4\pi t) = \cos(4\pi t + 2\pi) = \cos(4\pi(t + 1/2)),$$

and

$$\sin(80\pi t) = \sin(80\pi t + 2\pi) = \sin(80\pi(t + 1/40)),$$

we deduce that fixing $x = \bar{x}$ the function

$$t \mapsto u(\bar{x}, t),$$

is 1/2-periodic. The correct answer is the first one.