9.1. Separation of variables for non-homogeneous problems

Solve the following equations using the method of separation of variables and superposition principle. If the boundary conditions are non-homogeneous, find a suitable function satisfying the boundary conditions, and subtract it from the solution.

(a)

$$\begin{cases} u_t - u_{xx} &=& t + 2\cos(2x), & (x,t) \in (0,\pi/2) \times (0,\infty), \\ u_x(0,t) &=& 0, & t \in (0,\infty), \\ u_x(\pi/2,t) &=& 0, & t \in (0,\infty), \\ u(x,0) &=& 1 + 2\cos(6x), & x \in [0,\pi/2]. \end{cases}$$

Sol. In Lecture 9 we have seen that the solution of the inhomogeneous problem can be obtained by applying the method of separation of variables, that is writing

$$u(x,t) = \sum_{n\geq 0} T_n(t) X_n(x),$$

where the ODE solved by $X_n(x)$ and $T_n(t)$ are determined by the homogeneous problem (i.e. setting 0 to the right hand side of the PDE). We already know that the function $v(x,t) = X(x)T(t)$ solves the homogeneous problem $v_t - v_{xx} = 0$ if $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \lambda =$ constant, which implies as usual that

- If $\lambda > 0$,
$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$
- If $\lambda = 0$,
$$X(x) = A + Bx.$$
- If $\lambda < 0$,
$$X(x) = A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x).$$

Imposing the initial conditions $v_x(0,t) = v_x(\pi/2,t) = 0$ is equivalent to ask $X'(0) = X'(\pi/2) = 0$. This implies directly that $A = B = 0$ if $\lambda < 0$, and $B = 0$ if $\lambda = 0$. If $\lambda > 0$ we get that $B = 0$, and $\cos(\sqrt{\lambda}\pi/2) = 0$, that is, $\lambda_n = 4n^2$ is the set of possible values for λ (including $n = 0$ to account for the constant arising from $\lambda = 0$). Thus, the corresponding solutions are $X_n(x) = \cos(2nx)$, and we are looking for a general solution of the form

$$u(x,t) = \sum_{n\geq 0} T_n(t)\cos(2nx)$$

for some functions $T_n(t)$ to be determined. From the initial condition, we directly get that

$$T_0(0) = 1, \quad T_3(0) = 2, \quad T_n(0) = 0, \text{ for all } n \notin \{0,3\}.$$

On the other hand, imposing that $u_t - u_{xx} = t + 2\cos(2x)$ we get

$$\sum_{n\geq 0}(T_n'(t) + 4n^2 T_n(t))\cos(2nx) = t + 2\cos(2x).$$

That is,

$$T_0'(t) = t, \quad T_1'(t) + 4T_1(t) = 2, \quad T_n'(t) + 4n^2(t) = 0, \text{ for all } n \geq 2.$$

We have various ODE with the corresponding initial conditions to be solved for each T_n:

$(n = 0)$ In this case,

$$T_0'(t) = t, \quad T_0(0) = 1, \quad \Rightarrow \quad T_0(t) = 1 + \frac{1}{2}t^2.$$

$(n = 1)$ In this case,

$$T_1'(t) + 4T_1(t) = 2, \quad T_1(0) = 0, \quad \Rightarrow \quad T_1(t) = \frac{1}{2} - \frac{1}{2}e^{-4t}.$$

(To solve the ODE, we notice that the solution to the homogeneous ODE is Ce^{-4t}, and that a particular solution is simply the constant $\frac{1}{2}$. By adding them up, and choosing C such that the initial condition holds, we get our solution.)

$(n = 3)$ In this case,

$$T_3'(t) + 36T_3(t) = 0, \quad T_3(0) = 2, \quad \Rightarrow \quad T_3(t) = 2e^{-36t}.$$

$(n \notin \{0, 1, 3\})$ In this case,

$$T_n'(t) + 4n^2 T_n(t) = 0, \quad T_n(0) = 0, \quad \Rightarrow \quad T_n(t) = 0.$$

Thus, the general solution is given by

$$u(x,t) = 1 + \frac{1}{2}t^2 + \frac{1}{2}\left(1 - e^{-4t}\right)\cos(2x) + 2e^{-36t}\cos(6x).$$

(b)

$$\begin{cases} u_t - u_{xx} &= 1 + x\cos(t), & (x,t) \in (0,1) \times (0,\infty), \\ u_x(0,t) &= \sin(t), & t \in (0,\infty), \\ u_x(1,t) &= \sin(t), & t \in (0,\infty), \\ u(x,0) &= 1 + \cos(2\pi x), & x \in [0,1]. \end{cases}$$

Hint: The function $w(x,t) = x\sin(t)$ fulfills the boundary conditions from above.

Sol. The first thing to notice is that the boundary conditions are now non-homogeneous. Thus, we have to find a new function $w(x,t)$ satisfying such non-homogeneous boundary conditions, and study the problem being satisfied by $v(x,t) = u(x,t) - w(x,t)$.

In this case, from the hint $w(x,t) = x\sin(t)$ satisfies the boundary conditions for $t \geq 0$. Let us write the problem satisfied by $v(x,t) = u(x,t) - x\sin(t)$:

$$\begin{cases} v_t - v_{xx} &= 1, & (x,t) \in (0,1) \times (0,\infty), \\ v_x(0,t) &= 0, & t \in (0,\infty), \\ v_x(1,t) &= 0, & t \in (0,\infty), \\ v(x,0) &= 1 + \cos(2\pi x), & x \in [0,1]. \end{cases}$$

Solving the associated ODE problem coming from the separation of variables and imposing the boundary conditions as before, we get that the possible values of λ (the constant realising $\frac{X''}{X} = \frac{T'}{T} = \lambda$) are given by $\pi^2 n^2$ for $n \in \{0,1,2,\dots\}$, and the associated solutions are $X_n(x) = \cos(n\pi x)$. Thus, we are looking for a general solution of the form

$$v(x,t) = \sum_{n\geq 0} T_n(t)\cos(n\pi x).$$

From the initial condition, we directly get that

$$T_0(0) = 1, \quad T_2(0) = 1, \quad T_n(0) = 0, \text{ for all } n \notin \{0,2\}.$$

On the other hand, imposing that $v_t - v_{xx} = 1$ we get

$$\sum_{n\geq 0}(T_n'(t) + \pi^2 n^2 T_n(t))\cos(n\pi x) = 1.$$

That is,

$$T_0'(t) = 1, \quad T_n'(t) + \pi^2 n^2 T_n(t) = 0, \text{ for all } n \geq 1,$$

And we can solve the various ODE for each T_n:

$(n = 0)$ In this case,

$$T_0'(t) = 1, \quad T_0(0) = 1, \quad \Rightarrow \quad T_0(t) = 1 + t.$$

$(n = 2)$ In this case,

$$T_2'(t) + 4\pi^2 T_2(t) = 0, \quad T_2(0) = 1, \quad \Rightarrow \quad T_2(t) = e^{-4\pi^2 t}.$$

$(n \notin \{0,2\})$ In this case,

$$T_n'(t) + \pi^2 n^2 T_n(t) = 0, \quad T_n(0) = 0, \quad \Rightarrow \quad T_n(t) = 0.$$

Thus,

$$v(x,t) = 1 + t + e^{-4\pi^2 t}\cos(2\pi x),$$

and, therefore,

$$u(x,t) = v(x,t) + w(x,t) = 1 + t + e^{-4\pi^2 t}\cos(2\pi x) + x\sin(t).$$

(c) Mixed Boundary Conditions.

$$\begin{cases} u_t - u_{xx} &= \sin(9x/2), & (x,t) \in (0,\pi)\times(0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u_x(\pi,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= \sin(3x/2), & x \in [0,\pi]. \end{cases}$$

Sol. Once again, calling v the solution of the homogeneous equation $v_t - v_{xx} = 0$, $v(0,t) = v_x(0,t) = 0$ we have that the ODE obtained by the separation of variables $v(x,t) = X(x)T(t)$ has solutions

- If $\lambda > 0$,
$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$
- If $\lambda = 0$,
$$X(x) = A + Bx.$$
- If $\lambda < 0$,
$$X(x) = A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x).$$

If $\lambda = 0$, from $X(0) = 0$ we get that $A = 0$, and from $X'(\pi) = 0$ we get that $B = 0$, so that only the trivial solution remains.

If $\lambda < 0$, from $X(0) = 0$ we get $A = 0$, and from $X'(\pi) = 0$ we get $B = 0$, so that again, only the trivial solution remains.

Let $\lambda > 0$. From $X(0) = 0$ we get $A = 0$. From $X'(\pi) = 0$ we get $\cos(\sqrt{\lambda}\pi) = 0$. That is,

$$\sqrt{\lambda} = n + \frac{1}{2}, \quad \text{for} \quad n \in \{0,1,2,\dots\}.$$

Thus, the set of admissible values for λ is $\lambda_n = \left(n + \frac{1}{2}\right)^2$, and the corresponding solutions are $X_n(x) = \sin\left(\left(n + \frac{1}{2}\right)x\right)$.

We are looking for a general solution of the form

$$u(x,t) = \sum_{n\geq 0} T_n(t) \sin\left(\left(n+\frac{1}{2}\right)x\right)$$

for some functions $T_n(t)$ to be determined. From initial conditions,

$$T_1(0) = 1, \quad T_n(0) = 0, \text{ for all } n \neq 1.$$

Imposing that the equation is fulfilled, we get

$$u(x,t) = \sum_{n\geq 0} \left(T_n'(t) + \left(n+\frac{1}{2}\right)^2 T_n(t)\right) \sin\left(\left(n+\frac{1}{2}\right)x\right) = \sin\left(\frac{9x}{2}\right).$$

Thus, our ODEs are

$(n=1)$ In this case,

$$T_1'(t) + \frac{9}{4}T_1(t) = 0, \quad T_1(0) = 1, \quad \Rightarrow \quad T_1(t) = e^{-\frac{9t}{4}}.$$

$(n=4)$ In this case,

$$T_4'(t) + \frac{81}{4}T_4(t) = 1, \quad T_4(0) = 0, \quad \Rightarrow \quad T_4(t) = \frac{4}{81} - \frac{4}{81}e^{-\frac{81t}{4}}.$$

$(n \notin \{1,4\})$ In this case,

$$T_n'(t) + \left(n+\frac{1}{2}\right)^2 T_n(t) = 0, \quad T_n(0) = 0, \quad \Rightarrow \quad T_n(t) = 0.$$

And our solution is therefore given by

$$u(x,t) = e^{-\frac{9t}{4}} \sin\left(\frac{3x}{2}\right) + \frac{4}{81}\left(1 - e^{-\frac{81t}{4}}\right)\sin\left(\frac{9x}{2}\right).$$

(d)

$$\begin{cases} u_t - u_{xx} &= -u, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u(\pi,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= \sin(x), & x \in [0,\pi]. \end{cases}$$

Sol. In this case, the corresponding base of functions is given by $X_n(x) = \sin(nx)$ and $\lambda_n = n^2$ are the admissible values associated (see, for instance, Serie 8, Ex 8.1(a) where we have computed it in the homogeneous case). Thus, we are looking for a general solution of the form

$$u(x,t) = \sum_{n\geq 1} T_n(t)\sin(nx).$$

From the initial condition,

$$T_1(0) = 1, \quad T_n(0) = 0, \text{ for all } n \geq 2.$$

On the other hand, imposing that $u_t - u_{xx} + u = 0$,

$$\sum_{n\geq 1} (T_n'(t) + n^2 T_n(t) + T_n(t))\sin(nx) = 0.$$

That is,

$$T_n'(t) + (n^2+1)T_n(t) = 0, \quad \text{for all } n \geq 1.$$

Solving the corresponding ODEs,

$(n = 1)$ In this case,

$$T_1'(t) + 2T_1(t) = 0, \quad T_1(0) = 1, \quad \Rightarrow \quad T_1(t) = e^{-2t}.$$

$(n \geq 2)$ In this case,

$$T_n'(t) + (n^2+1)T_n(t) = 0, \quad T_n(0) = 0, \quad \Rightarrow \quad T_n(t) = 0.$$

And the general solution is given by

$$u(x,t) = e^{-2t}\sin(x).$$