10.1. Unique solution

Let $k > 0$ be a positive constant. Let D be a bounded planar domain in $\mathbb{R}^2$ (that is, $(x, y) \in D \subset \mathbb{R}^2$), and let $g = g(x, y)$ be a continuous function defined on the boundary ∂D (that is, $g \in C(\partial\Omega)$). Let $u = u(x, y)$ be a solution to the Dirichlet problem for the reduced Helmholtz energy in D: let u solve

$$\begin{cases} \Delta u(x,y) - ku(x,y) &= 0, \quad \text{for } (x,y) \in D, \\ u(x,y) &= g(x,y), \quad \text{for } (x,y) \in \partial D. \end{cases}$$

Show that there exists at most a unique solution twice differentiable in D and continuous in $\overline{D}$, that is, $u \in C^2(D) \cap C(\overline{D})$.

Hint: Assume that there exist two solutions u_1 and u_2, and consider the difference $v = u_1 - u_2$. Arguing as in the proof of the weak maximum principle for the Laplace equation (Theorem 7.5 and Remark 7.6 from Pinchover's book), show that $\max_{\overline{D}} v = \max_{\partial D} v$ and $\min_{\overline{D}} v = \min_{\partial D} v$. Then, use this information to conclude that $u_1 = u_2$.

Sol. Let us use the hint. Let us suppose that there exist two solutions u_1 and u_2 fulfilling the Dirichlet problem. Let $v = u_1 - u_2$. Notice that $v = v(x, y)$ solves

$$\begin{cases} \Delta v(x,y) - kv(x,y) &= 0, \quad \text{for } (x,y) \in D, \\ v(x,y) &= 0, \quad \text{for } (x,y) \in \partial D. \end{cases}$$

We just need to prove that $v \equiv 0$ in D. To do so, we will show that $\max_{\overline{D}} v = \min_{\overline{D}} v = 0$. We show both equalities by contradiction.

Notice that $\max_{\overline{D}} v \geq 0$, since $v = 0$ on ∂D. Let us suppose that $\max_{\overline{D}} v = M > 0$. In particular, there exists some $(x_\circ, y_\circ) \in D$ such that $v(x_\circ, y_\circ) = M > 0$, that is, v has a maximum at $(x_\circ, y_\circ)$. In particular, we know that $\Delta v(x_\circ, y_\circ) \leq 0$. Therefore,

$$0 = \Delta v(x_\circ, y_\circ) - kv(x_\circ, y_\circ) \leq -kM < 0,$$

a contradiction.

On the other hand, $\min_{\overline{D}} v \leq 0$, since $v = 0$ on ∂D. Let us suppose that $\min_{\overline{D}} v = m < 0$. In particular, there exists some $(x_\circ, y_\circ) \in D$ such that $v(x_\circ, y_\circ) = m < 0$, that is, v has a minimum at $(x_\circ, y_\circ)$. In particular, we know that $\Delta v(x_\circ, y_\circ) \geq 0$. Therefore,

$$0 = \Delta v(x_\circ, y_\circ) - kv(x_\circ, y_\circ) \geq -km > 0,$$

a contradiction. Therefore, if there exists a solution, is unique.

10.2. The mean-value principle Let D be a planar domain, and let $B_R((x_\circ, y_\circ))$ (ball of radius R centered at $(x_\circ, y_\circ)$) be fully contained in D. Let u be an harmonic function in D, $\Delta u = 0$ in D. Then, the mean-value principle says that the value of u at $(x_\circ, y_\circ)$ is the average value of u on $\partial B_R((x_\circ, y_\circ))$. That is,

$$u(x_\circ, y_\circ) = \frac{1}{2\pi R}\oint_{\partial B_R((x_\circ,y_\circ))} u(x(s), y(s))\, ds = \frac{1}{2\pi}\int_0^{2\pi} u(x_\circ + R\cos\theta, y_\circ + R\sin\theta)\, d\theta.$$

Show that $u(x_\circ, y_\circ)$ is also equal to the average of u in $B_R((x_\circ, y_\circ))$, that is,

$$u(x_\circ, y_\circ) = \frac{1}{\pi R^2}\int_{B_R((x_\circ,y_\circ))} u(x, y)\, dx\, dy.$$

Sol. Let us use polar coordinates to compute

$$\begin{aligned}
\frac{1}{\pi R^2}\int_{B_R((x_\circ,y_\circ))} u(x, y)\, dx\, dy &= \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} u(x_\circ + r\cos\theta, y_\circ + r\sin\theta) r\, d\theta\, dr \\
&= \frac{1}{\pi R^2}\int_0^R r\left(\int_0^{2\pi} u(x_\circ + r\cos\theta, y_\circ + r\sin\theta)\, d\theta\right) dr \\
&= \frac{1}{\pi R^2}\int_0^R 2\pi r u(x_\circ, y_\circ)\, dr \\
&= u(x_\circ, y_\circ)\frac{1}{\pi R^2}[\pi r^2]_0^R \\
&= u(x_\circ, y_\circ).
\end{aligned}$$

We have used here the boundary mean value principle in the balls $B_r((x_\circ, y_\circ))$ for each $r \in (0, R)$.

10.3. Weak maximum principle Let B_1 denote the unit ball in $\mathbb{R}^2$ centered at the origin, and let $u = u(x, y)$ be twice differentiable in B_1 and continuous in $\overline{B_1}$ (that is, $u \in C^2(B_1) \cap C(\overline{B_1})$). Suppose that u solves the Dirichlet problem

$$\begin{cases} \Delta u(x, y) = -1, & \text{for } (x, y) \in B_1, \\ u(x, y) = g(x, y), & \text{for } (x, y) \in \partial B_1. \end{cases}$$

Show that

$$\max_{\bar{B}_1} u \leq \frac{1}{2} + \max_{\partial B_1} g.$$

Hint: search for a simple *function w such that $\Delta w = 1$, and use it to reduce the problem to an application of the weak maximum principle for harmonic functions.*

Sol. We just need to find a function $w(x, y)$ such that $\Delta w(x, y) = 1$, and then consider $v(x, y) = u(x, y) + w(x, y)$. The simplest function such that $\Delta w(x, y) = 1$ is $w(x, y) = \frac{1}{2}x^2$. Thus, let us define

$$v(x, y) = u(x, y) + \frac{1}{2}x^2.$$

Then, v solves

$$\begin{cases} \Delta v(x,y) = 0, & \text{for } (x,y) \in B_1, \\ v(x,y) = g(x,y) + \frac{1}{2}x^2, & \text{for } (x,y) \in \partial B_1. \end{cases}$$

By the weak maximum principle, we know that

$$\max_{\bar{B}_1} v(x,y) = \max_{\partial B_1} \left(g(x,y) + \frac{1}{2}x^2 \right) \leq \max_{\partial B_1} g(x,y) + \max_{\partial B_1} \frac{1}{2}x^2.$$

Notice that $\max_{\partial B_1} \frac{1}{2}x^2 = \frac{1}{2}$, so

$$\max_{\bar{B}_1} v(x,y) \leq \frac{1}{2} + \max_{\partial B_1} g(x,y).$$

On the other hand, $v(x,y) \geq u(x,y)$ for all $x, y \in B_1$, so

$$\max_{\bar{B}_1} u(x,y) \leq \max_{\bar{B}_1} v(x,y) \leq \frac{1}{2} + \max_{\partial B_1} g(x,y),$$

as we wanted to see.

10.4. Multiple Choice Determine the correct answer.

(a) Consider the Neumann problem for the Poisson equation

$$\begin{cases} \Delta u = \rho, & \text{in } D, \\ \partial_\nu u = g, & \text{on } \partial D, \end{cases}$$

where $D = B(0,R)$ is the ball of radius $R > 0$ with centre in the origin of $\mathbb{R}^2$, and ρ and g are given in polar coordinates (r,θ) by

$$\rho(r,\theta) = r^\alpha \sin^2(\theta), \text{ and } g(r,\theta) = C\cos^2(\theta) + r^{2021}\sin(\theta),$$

for some constants $\alpha > 0$ and $C > 0$. For which values of $C > 0$ does the problem satisfy the Neumann's *necessary* condition for existence of solutions?

- ☐ $C = \frac{R^{\alpha+1}}{\alpha+2}$
- ☐ $C = \frac{R^{\alpha+1}}{\alpha+1}$
- ☐ $C = \frac{R^{\alpha+2}}{\alpha+2}$

Sol. We say that the Neumann Problem for the Poisson equation satisfies the necessary condition for existence of solutions if the identity

$$\int_{\partial D} g = \int_D \rho, \tag{1}$$

holds. In our particular case we can compute in polar coordinates

$$\int_D \rho = \int_0^R r \int_0^{2\pi} r^\alpha \sin^2(\theta)\, d\theta\, dr = \pi \frac{R^{\alpha+2}}{\alpha+2},$$

and parametrizing ∂D with the curve $\theta \mapsto (R\cos(\theta), R\sin(\theta))$ we have that

$$\int_{\partial D} g = \int_0^{2\pi} R\Big(C\cos^2(\theta) + R^{2021}\sin(\theta)\Big)\, d\theta = RC\pi.$$

Plugging this in Equation (1) we obtain that the identity

$$RC\pi = \pi \frac{R^{\alpha+2}}{\alpha+2},$$

is valid if and only if $C = \frac{R^{\alpha+1}}{\alpha+2}$. The correct answer is the first one.