Notes Analysis 3 - ETH Zürich

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January 4, 2022

Contents

Notation

$\mathbf{x}$ The variables in bold denote vectors, e.g. $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

C^k A function is C^k if it is continuously differentiable k times.

CHAPTER 1

PRELIMINARIES

1.1. Partial differential equations

Definition 1.1.1. An *ordinary differential equation*, or *ODE*, is an equation involving functions of one independent variable and one or more of their derivatives.

Example 1.1.2. An example of ODE is Newton's second law, that is

$$m\frac{\mathrm{d}^2\mathbf{x}(t)}{\mathrm{d}t^2} = F(\mathbf{x}(t))\,.$$

The unknown function $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^3$ represents the position of a particle at time t. Moreover $m \in \mathbb{R}^+$ is the mass of the particle and $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the force field. Note that

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} \qquad \text{and} \qquad \frac{\mathrm{d}^2\mathbf{x}(t)}{\mathrm{d}t^2}$$

represent respectively the velocity and the acceleration of the particle.

Definition 1.1.3. A *partial differential equation*, or *PDE*, is an equation involving an unknown function of more than one variable and certain of its partial derivatives.

Hereafter u denotes the real-valued solution (i.e., the unknown) of a given PDE and it is usually a function of points $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, typically denoting a position in space. Sometimes the function u also depends on a parameter $t \in \mathbb{R}$, denoting the time. We also use x, y, z to denote independent variables (instead of $x_1, x_2, x_3, \ldots$).

Notation. We write

$$u_{x_k} = \frac{\partial u}{\partial x_k}$$

to denote the partial derivative of u with respect to x_k. Analogously we use

$$u_t = \frac{\partial u}{\partial t} \qquad \text{and} \qquad u_{x_k x_l} = \frac{\partial^2 u}{\partial x_k \partial x_l}\,.$$

We recall the following theorem.

Theorem 1.1.4 (Schwartz). *Given a function u continuously differentiable at a point, the order of partial derivatives at that point is irrelevant. Namely, if u is continuously differential at a point $\mathbf{x}$ and u depends on two variables x, y, then $u_{xy}(\mathbf{x}) = u_{yx}(\mathbf{x})$.*

Definition 1.1.5. The *gradient* of a function $u = u(x, y, z)$ is defined as

$$\nabla u := (u_x, u_y, u_z)$$

and the *Laplacian* of u is

$$\Delta u := u_{xx} + u_{yy} + u_{zz} \,.$$

We now see some examples that motivate the use of partial differential equations: when we want to study a physical system we want to understand the state of such system at any point in space and at any time.

Example 1.1.6. Suppose that $u(x, y, z, t)$ is the temperature at the point (x, y, z) and at time t. We know that this state changes over time, so we consider the quantity u_t, which measures this change with respect to time. However, u also changes with respect to the position, so we consider partial derivatives of u with respect to x, y, z. Surely we need to relate the variations in space and in time and it turns out that the heat flow over time may be described by

$$u_t = u_{xx} + u_{yy} + u_{zz} = \Delta u \,. \tag{Heat equation}$$

Example 1.1.7. Another fundamental equation that we will encounter is the Laplace's equation, which records diffusion effects in equilibrium and it is described by the PDE

$$\Delta u = 0 \,. \tag{Laplace's equation}$$

Example 1.1.8. The wave equation, given by

$$u_{tt} = c^2 \Delta u \,, \tag{Wave equation}$$

superficially resembles the heat equation, but it supports solutions with a completely different behaviour and can be used to describe the propagation of a wave in a fluid.

Example 1.1.9. The Burgers' equation

$$u_t = u u_x \tag{Burgers' equation}$$

can model the flow of a viscous fluid or the traffic flow, and it is a prototypical example of conservation law (see Chapter 3).

1.2. What is a well-posed problem?

In general we study equations that originate from a physical or engineering problem, so we follow the scheme

$$\text{real life problem} \quad \rightsquigarrow \quad \text{model} \quad \rightsquigarrow \quad \text{PDE}.$$

It is not obvious that a given model is consistent, in the sense that it leads to a solvable PDE. Furthermore we wish the solution to be unique and to be stable under small perturbation of the data.

By "problem" we mean a PDE supplemented with initial or boundary conditions. A problem is *well-posed* if it satisfies the following criteria:

1. The problem has a solution (*existence*).
2. The solution is unique (*uniqueness*).
3. A small change in the equation and/or in the side conditions gives rise to a small change in the solution (*stability*).

If one or more of these conditions do not hold, then the problem is said to be *ill-posed*.

1.3. Initial and boundary conditions

As you may recall from studying ODEs, there may be no solutions or there may be many solutions for a given ODE (if the initial conditions are not appropriate). The same is true for PDEs. Indeed PDEs have in general infinitely many solutions and in order to obtain a unique solution we must supplement the equation with additional conditions. What kind of conditions? This depends on the type of PDE under study. We will focus entirely on PDEs coupled with a set of initial conditions consisting in prescribing the unknown u and (or) some of its partial derivatives on an hypersurface of the domain. In $\mathbb{R}^2$ this simply means that we fix some initial data on a curve, and in $\mathbb{R}^3$ on a surface (which will be a plane most of the time). This is what we call a *Cauchy problem*

Example 1.3.1. Consider the transport equation

$$u_t + cu_x = 0\,, \qquad \text{(Transport equation)}$$

for $u\colon (x,t) \in \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ and $c \in \mathbb{R}$ a constant. If $u \geq 0$, u can represent the concentration of a pollutant in a river at time t and position x. The constant $c \in \mathbb{R}$ represents the velocity of the river. In order to have a complete information about u in time it makes sense to couple this equation with an information about the concentration of the pollutant at time zero. Hence we consider the *initial value problem*

$$\begin{cases} u_t + cu_x = 0\,, & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = g(x)\,, & x \in \mathbb{R}\,, \end{cases}$$

where the function $g > 0$ represents the concentration of the pollutant at time zero.

Example 1.3.2. In the case of the one-dimensional vibrating string, we consider the wave equation with four side condition, namely

$$\begin{cases} u_{tt} = u_{xx}\,, & (x,t) \in (0,L) \times \mathbb{R}^+ \\ u(0,t) = u(L,t) = 0\,, & t \geq 0 \\ u(x,0) = f(x)\,, & x \in [0,L] \\ u_t(x,0) = g(x)\,, & x \in [0,L]\,. \end{cases}$$

The second equation expresses two *boundary conditions* (the string is fixed at position 0 and L) and the last two equations express the *initial conditions*: they tell us what happens at time zero in terms of the deflection $f(x)$ and of the speed $g(x)$.

Remark 1.3.3. The domain of the PDE is defined only in the interior of the interval because the function u may not be differentiable on the boundary.

Definition 1.3.4. We say that the solution of a PDE is *strong* if all the derivatives of the solution that appear in the PDE exist and are continuous. Otherwise the solution is said to be *weak*.

Weak solutions have points in their domain where the derivatives do not exist (or are not continuous), so a weak solution cannot directly be plugged into the equation.

Remark 1.3.5. There is no universal meaning for weak solution, the definition depends on the type of PDE and we will see this later when studying conservation laws.

1.4. Classification properties of PDEs

Definition 1.4.1. The *order* of a PDE is the order of the highest order partial derivative of the unknown appearing within it.

Example 1.4.2.

- $u_x + u^2 u_y y = e^x u_{xy}$ has order 2;

- $u_{xyz} = xy^2 + zu$ has order 3;
- $u_{tt} = u_{xx} + f(x,t)$ has order 2.

Remark 1.4.3. Here we mostly work with PDEs of first and second order.

Definition 1.4.4. A PDE is *linear* if it is of the form

$$a^{(0)}u + \sum_{i_1=1}^{n} a_{i_1}^{(1)} u_{x_{i_1}} + \sum_{i_1,i_2=1}^{n} a_{i_1,i_2}^{(2)} u_{x_{i_1} x_{i_2}} + \ldots =: \mathcal{L}[u] = f(\mathbf{x}) \,, \tag{1.4.1}$$

where $f(\mathbf{x})$ and $a_{i_1,\ldots,i_m}^{(m)}$ are functions of the variable $\mathbf{x} = (x_1, \ldots, x_n)$. Namely, a PDE is linear if every summand consists of a function multiplied by u or by one of its derivatives. Equivalently, if u and v solves (1.4.1), then $u - v$ solves (1.4.1) with $f = 0$.

Remark 1.4.5. Observe that the general form of a linear PDE of the first order for an unknown function u in two independent variables x, y is

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \,,$$

while the general form of a linear PDE of the 2nd order is

$$a(x,y)u_{xx} + b(x,y)u_{yy} + 2c(x,y)u_{xy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y) \,.$$

Example 1.4.6.

- $xyu_x + \sin^2(y)u_{xy} - e^x u_{yy} = 2xy^3$ is linear;
- $uu_x = 2$ is not linear;
- $u_t = u_x + u^2$ is not linear;
- $u_{tt} = u_{xxxx}$ is linear.

Definition 1.4.7. We say that a linear PDE of the form defined in (1.4.1) is *homogeneous* if $f(\mathbf{x}) = 0$. When $f \not\equiv 0$, we say that the PDE is *inhomogeneous* and the function $f(\mathbf{x})$ is the *inhomogeneity*.

An important property of linear homogeneous PDEs is that, given two solutions u_1 and u_2, any linear combination of u_1 and u_2 is a solution as well. Moreover, solutions of a linear homogeneous PDEs generate different solutions of an associated inhomogeneous PDE. Let us see how in the following theorem.

Theorem 1.4.8. *Let $\mathcal{L}[u] = f(\mathbf{x})$ be a linear inhomogeneous PDE and $\mathcal{L}[u] = 0$ be the corresponding homogeneous PDE. Let u_1, u_2 be solutions of $\mathcal{L}[u] = 0$ and u_p be a solution of $\mathcal{L}[u] = f(\mathbf{x})$. Then, for all $\alpha, \beta \in \mathbb{R}$, we have that $\alpha u_1 + \beta u_2$ is a solution of $\mathcal{L}[u] = 0$ and $\alpha u_1 + \beta u_2 + u_p$ is a solution of $\mathcal{L}[u] = f(\mathbf{x})$.*

Remark 1.4.9. We can denote with $\mathcal{L}[u]$ any linear operator acting on u, as in (1.4.1). For example, given the linear operator $\mathcal{L}[u] = u_t - u_x$, the transport equation can be written as $\mathcal{L}[u] = 0$.

Example 1.4.10.

- $u_t + u_x = 0$ is linear, homogeneous;
- $u_{xx} + u_{yy} = x^2 + y^2$ is linear, inhomogeneous;
- $u_x^2 + u_y^2 = 1$ is nonlinear.

As you may be able to guess, many PDEs are not linear and nonlinear equations are often further classified into subclasses according to the type of nonlinearity. We will sometimes handle nonlinear PDEs that still have a special structure called *quasilinearity*. The main reason to make all these distinctions lies in the tools available to solve each type of equation. For example, the *method of characteristics* allow us to solve first order quasilinear PDEs.

Definition 1.4.11. We say that a PDE is *quasilinear* if it is linear in its highest order derivative term.

Remark 1.4.12. The general form of a quasilinear PDE of the first order for an unknown $u(x, y)$ depending on two variables is

$$a(x, y, u)u_x + b(x, y, u)u_y + c(x, y, u) = 0\,.$$

Note that the functions a, b, c may depend also on u but not on u_x, u_y. The general form of a quasilinear PDE of the 2nd order is instead

$$a(x, y, u, u_x, u_y)u_{xx}+b(x, y, u, u_x, u_y)u_{yy}+2c(x, y, u, u_x, u_y)u_{xy}+d(x, y, u, u_x, u_y) = 0\,.$$

Example 1.4.13.

- $uu_x + x^2u_y = u^3$ is quasilinear, of the first order;
- $u_x + uu_y = 0$ is quasilinear, of the first order;
- $uu_x + u^2u_y + u = e^4$ is quasilinear, of the first order;
- $u_xu_y = x^2y$ is not quasilinear, but is of the first order;
- $u_t = u_yu_{xx} + u^2u_{yy} + u_x^2$ is quasilinear, of the second order;
- $u_{xy}^2 = xu + u_y$ is not quasilinear and it is of the second order.

Remark 1.4.14. A nonlinear equation can be quasilinear if it is linear in its highest order terms. A quasilinear equation is a nonlinear equation but with the good property of being linear with respect to the highest order terms, which can be thought as “dominant”.

Example 1.4.15.
- $u_x + u_{tt} + 2u = e^x$ is of the 2nd order, nonhomogeneous and linear;
- $u_x + u_{tt} + 3uu_x = e^x$ is of the 2nd order, nonlinear, but quasilinear (u_{tt} appears linearly);
- $u_{xx} + (u_t)^2 + e^u = 0$ is of the 2nd order, nonlinear, but quasilinear;
- $(u_{xx})^2 + u_t + e^u = 0$ is of the 2nd order and nonlinear, because u_{xx} appears nonlinearly;
- $\sin(u) + u_x + u_{yy} = 0$ is of the 2nd order and quasilinear;
- $u + u_x + \sin(u_{yy}) = 0$ is of the 2nd order, nonlinear;
- $u\sin(u_x) + u_{yy} = 0$ is of the 2nd order, nonlinear, but quasilinear;
- $(u_x)^2 + (u_y)^2 + u_{xy} = 0$ is of the 2nd order, quasilinear;
- the Monge-Ampère equation $\det(D^2u) = f$, (where D^2u denotes the Hessian matrix $(D^2u)_{ij} := u_{x_ix_j}$) is of 2nd order, nonlinear, since the determinant is multilinear and linear only for 1×1 matrices;
- $|\nabla u| = f$ is of the first order, nonlinear;
- $u_t + \operatorname{div}(\vec{v}u) = \Delta u + f$ is of the 2nd order, nonhomogeneous and linear for any given constant vector $\vec{v} = (v_1, \dots, v_n)$.

Remark 1.4.16. Divergence, Laplacian and gradient are linear operators.

1.5. Modelling a stock market

We conclude this introduction with a modelling example.

Example 1.5.1. Let us model a stock market as follows:

$$\begin{cases} Y(t) = \text{ price of an asset} \\ Y(0) = 1\,. \end{cases}$$

Asset prices grow and decay exponentially, so we prefer to look at

$$\begin{cases} X(t) = \log(Y(t)) - rt \\ X(0) = 0\,, \end{cases}$$

where $r > 0$ is the growth rate. We now analyze the evolution of $X(t)$ using the *Merton model*. We assume that, given a time step $\tau > 0$, the value at time $t + \tau$ is given by $X(t + \tau) = X(t) \pm \delta$, for some $\delta > 0$. We choose to add or subtract δ with probability $1/2$ each. Let us define $p(x, t) := \text{Prob}(X(t) = x)$. Then the equation for $X(t + \tau)$ gives

$$p(x, t + \tau) = \frac{1}{2} p(x + \delta, t) + \frac{1}{2} p(x - \delta, t) \,.$$

Rearranging the terms, we thus obtain that

$$\frac{p(x, t + \tau) - p(x, t)}{\tau} = \frac{\delta^2}{2\tau} \cdot \frac{p(x + \delta, t) + p(x - \delta, t) - 2p(x, t)}{\delta^2} \,.$$

Assuming that $\delta = \sqrt{2k\tau}$ for some $k > 0$ and taking the limit $\tau \to 0$, we get

$$p_t(x, t) = k p_{xx}(x, t) \,.$$

As a result, the probability density p must fulfill the heat equation, whose solution is well-known and it is given by

$$p(x, t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4\pi k t}} \,.$$

Summary

- PDEs are normally used together with boundary conditions.
- Well-posedness stands for the existence and uniqueness of a stable solution.
- Linearity stands for linear in u, while quasilinear means linear in the highest order derivatives of u.
- A linear PDE is homogeneous if all the terms depend linearly on u.

CHAPTER 2

METHOD OF CHARACTERISTICS

In this chapter we present an approach to solve first order quasilinear PDEs known as *method of characteristics*. This method relies on a powerful geometrical interpretation of first order PDEs, reducing them to a system of ODEs.

2.1. First order equations

A first order PDE for an unknown function $u(x_1, \ldots, x_n)$ can be written in general form as

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = 0\,, \tag{2.1.1}$$

where F is a given function of $2n + 1$ variables. For our purposes, we consider two-dimensional real-valued function $u(x, y)$ for which the equation (2.1.1) reduces to

$$F(x, y, u, u_x, u_y) = 0\,. \tag{2.1.2}$$

Given a solution u to this equation, the graph of u, defined as

$$\operatorname{graph}(u) := \{(x, y, u(x, y)) \in \mathbb{R}^3\},$$

is the *solution surface* and it is indeed a surface in $\mathbb{R}^3$ with normal vector at a point $(x, y, u(x, y))$ given by $(u_x, u_y, -1)$ (see Figure 2.1). Hence, observe that equation (2.1.2) relates the graph surface to its normal, or equivalently to its tangent plane, which is the plane orthogonal to the normal. In fact, the tangent plane at each point of the surface $\operatorname{graph}(u)$ is the plane spanned by the vectors $(1, 0, u_x)$ and $(0, 1, u_y)$. Indeed note that $(1, 0, u_x)$ and $(0, 1, u_y)$ are linearly independent and orthogonal to the normal $(u_x, u_y, -1)$. See Figure 2.2 for a representation in one dimension less.

The point of this discussion is that a first order PDE can be seen geometrically as a relation between the solution surface and its tangent plane.

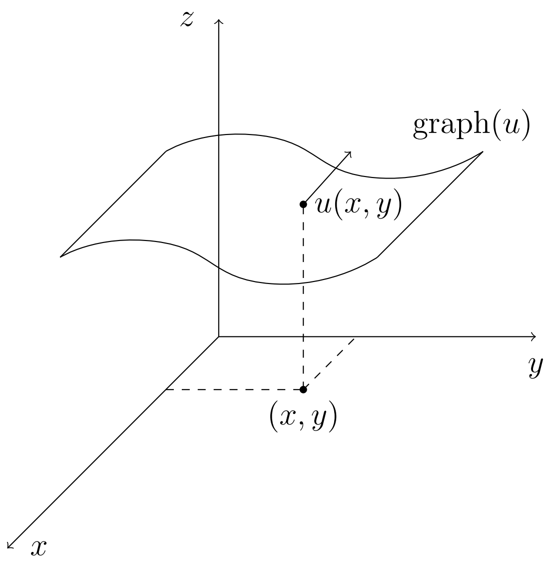

Figure 2.1: Graph of the solution surface.

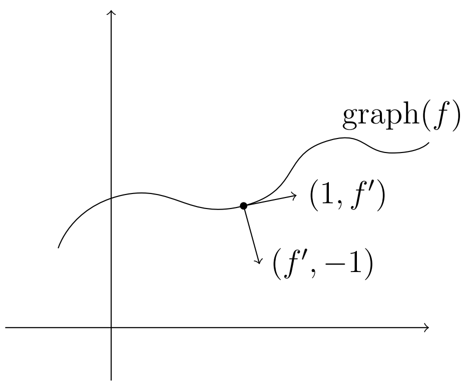

Figure 2.2: The tangent to the graph of a function $f\colon \mathbb{R} \to \mathbb{R}$ is given by the vector $(1, f')$.

2.2. Quasilinear equations

First order quasilinear equations are nonlinear PDEs, where the nonlinearity is confined to the unknown function u, while the derivatives of u appear linearly (see Definition 1.4.11). Thus, the general form of a first order quasilinear PDE (in two variables) is

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)\,, \tag{2.2.1}$$

where a, b, c are functions of three variables.

Example 2.2.1. Consider the first order PDE

$$u_x(x, y) = c_0 u(x, y) + c_1(x, y)\,,$$

where $c_0 \in \mathbb{R}$ is a constant. Fixing $y \in \mathbb{R}$, this becomes a first order ODE, which can be written as

$$[u_x(x, y) - c_0 u(x, y)]\, e^{-c_0 x} = c_1(x, y)e^{-c_0 x}\,,$$

or equivalently as

$$\frac{\partial}{\partial x}\left(u(x, y)e^{-c_0 x}\right) = c_1(x, y)e^{-c_0 x}\,.$$

Integrating both sides over an interval of the form $[x_0(y), x]$, we have

$$u(x, y)e^{-c_0 x} - u(x_0(y), y)e^{-c_0 x_0(y)} = \int_{x_0(y)}^{x} c_1(\xi, y)e^{-c_0 \xi}\,\mathrm{d}\xi\,.$$

Therefore we obtain that

$$u(x, y) = e^{c_0 x}\left[u(x_0(y), y)e^{-c_0 x_0(y)} + \int_{x_0(y)}^{x} c_1(\xi, y)e^{-c_0 \xi}\,\mathrm{d}\xi\right]\,. \tag{2.2.2}$$

This means that once we prescribe the value of u on the curve $\{(x_0(y), y) \,:\, y \in \mathbb{R}\}$, we can reconstruct the value of u everywhere (see Figure 2.3).

Depending on the initial conditions, we may have one solution, no solutions or infinitely many solutions. Let us look into some specific examples.

- Let us require as initial condition $u(0, y) = y$ for all $y \in \mathbb{R}$. Then we can set $x_0(y) = 0$, for which $u(x_0(y), y) = y$, and the solution of the equation is given by

$$u(x, y) = e^{c_0 x}\left[y + \int_0^x c_1(\xi, y)e^{-c_0 \xi}\,\mathrm{d}\xi\right]\,.$$

  Hence the solution is unique (see Figure 2.4).

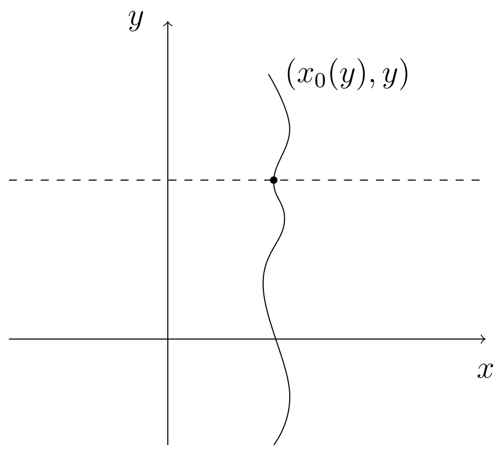

Figure 2.3: Once we prescribe the value of u on the curve $(x_0(y), y)$, for example at the intersection with the dotted line, we can reconstruct the value of u everywhere on the dotted line.

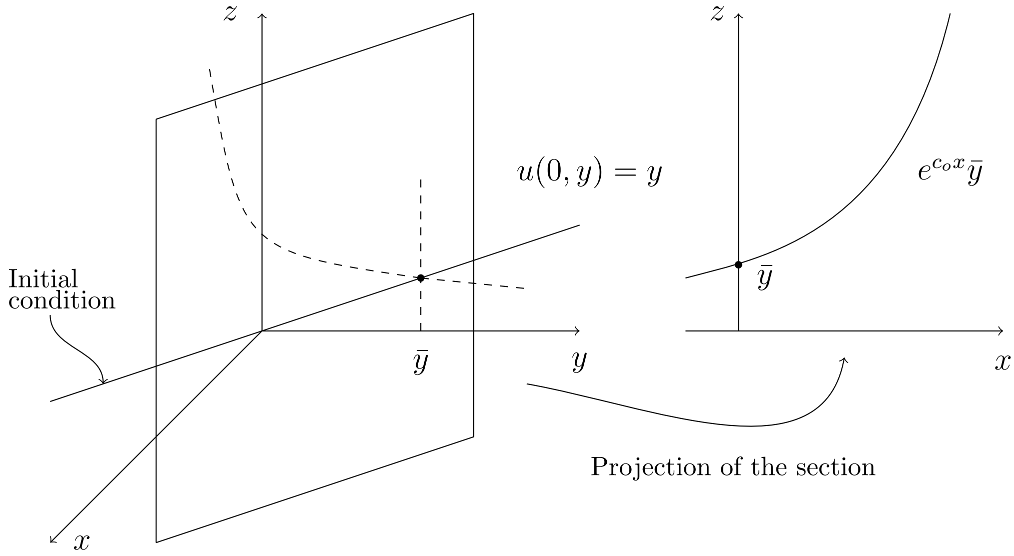

Figure 2.4: Figures illustrating the case of one solution in three dimensions. Here the initial datum is just one point in the plane $\{y = \bar{y}\}$, which implies both existence and uniqueness of the solution.

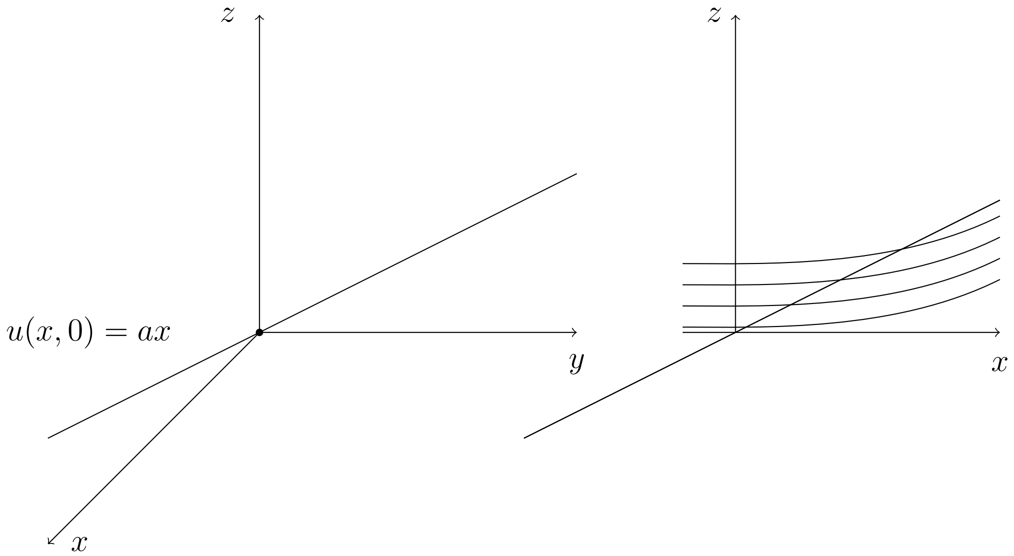

Figure 2.5: The exponential curves of the right figure are what the PDE wants the solution to be. For this reason, in this case we have no solution at all.

- Let us assume that $c_1 \equiv 0$, so that the general solution is given by

$$u(x,y) = e^{c_0 x}\left[u(x_0(y),y)e^{-c_0 x_0(y)}\right] = e^{c_0 x}T(y)\,,$$

  where $T(y) := u(x_0(y),y)e^{-c_0 x_0(y)}$. If we prescribe $u(x,0) = ax$ as initial condition for a nonzero constant $a \in \mathbb{R}$, then $T(y)$ has to satisfy $T(0) = u(x,0)e^{-c_0 x} = axe^{-c_0 x}$ for all $x \in \mathbb{R}$, which is patently impossible (see Figure 2.5).

- Assume as before that $c_1 \equiv 0$, but now require as initial condition $u(x,0) = e^{c_0 x}$ for all $x \in \mathbb{R}$. Plugging $y = 0$ in (2.2.2), we then obtain

$$e^{c_0 x} = u(x,0) = e^{c_0 x}u(x_0(0),0)e^{-c_0 x_0(0)}\,.$$

  Thus we only need to impose that $u(x_0(0),0)e^{-c_0 x_0(0)} = 1$. In particular, we can take any curve $y \mapsto (x_0(y),y)$ and choose the value of u on this curve as we want, provided that $u(x_0(0),0) = e^{c_0 x_0(0)}$. For example, we can consider the curve $y \mapsto (0,y)$ and set $x_0(0) = 0$, $u(0,y) = 1 + Ay^2$. Then

$$u(x,y) = e^{c_0 x}(1 + Ay^2)$$

  is a solution for every $A \in \mathbb{R}$ (see Figure 2.6).

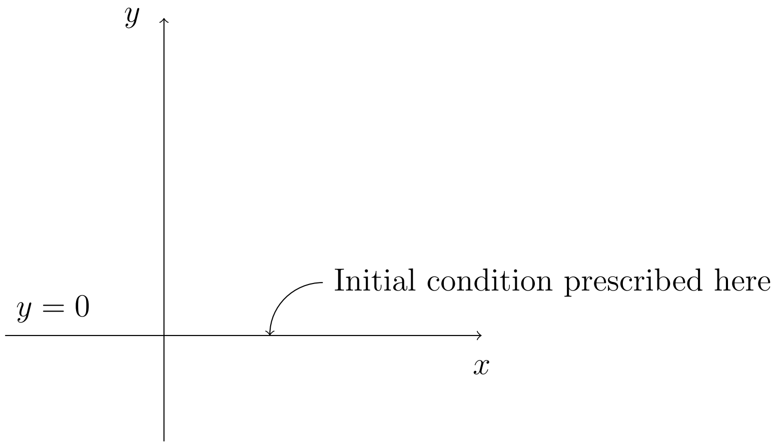

Figure 2.6: Once we prescribe the value of u at one point on the curve $\{y = 0\}$ then we obtain the value of u at all other points on the curve $\{y = 0\}$. If the value prescribe at $\{y = 0\}$ is compatible then we have infinitely many solutions, otherwise no solutions.

To summarize, if we consider the PDE

$$u_x(x, y) = c_0 u(x, y)$$

for a constant $c_0 \in \mathbb{R}$, then the general formula for a solution is

$$u(x, y) = e^{c_0 x} u(x_0(y), y) e^{-c_0 x_0(y)} .$$

So for all $y \in \mathbb{R}$ we take $x_0(y) \in \mathbb{R}$, defining a curve $y \mapsto (x_0(y), y)$. Then the value of u at each point $(x_0(y), y)$ defines u along all the horizontal line passing through that point.

- In the first example the initial condition was $y = u(0, y)$ and we have a *unique solution.*
- In the second case, the condition $u(x, 0) = ax$ for a constant $a \neq 0$ is not compatible with the PDE and we have *no solutions.*
- In the third example the initial condition $u(x, 0) = e^{c_0 x}$ is compatible with the PDE but it leaves "too much choice". In fact we have *infinitely many solutions* of the PDE.

The moral of the story is that boundary conditions and initial conditions are very important. We need to be careful to impose appropriate conditions in order to obtain a well-posed PDE.

2.3. Method for first order linear PDEs

Consider a general first order linear equation in two independent variables, namely

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = c_0(x,y)u(x,y) + c_1(x,y)\,. \tag{2.3.1}$$

The idea is to assign the value of the solution u along a parametric curve and then "propagate" this value along "characteristic curves".

So, given a curve $s \mapsto (x_0(s), y_0(s))$, we prescribe the value of u along such curve as

$$u(x_0(s), y_0(s)) = \tilde{u}_0(s)$$

for all $s \in \mathbb{R}$, for some function $\tilde{u}_0$ of one variable. Hence, if we obtain a solution u, the parametric curve in $\mathbb{R}^3$

$$\Gamma = \Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s))$$

is contained in graph(u). We say that Γ is the *initial curve.*

Now observe that equation (2.3.1) can be rewritten as

$$(a, b, c_0 u + c_1) \cdot (u_x, u_y, -1) = 0\,.$$

Namely we ask the vector $\vec{v} := (a, b, c_0 u + c_1)$ to be orthogonal to the normal vector $(u_x, u_y, -1)$ and thus tangent to the surface graph(u). As a result, if we integrate the vector field $\vec{v}$, i.e., we consider the ODE $\dot{\vec{z}} = \vec{v}(\vec{z})$, then the curve $\vec{z}$ is contained in the surface graph(u) (see Figure 2.7).

In other words, to find a solution to (2.3.1), we look for a surface $S \subseteq \mathbb{R}^3$ (which will then be graph(u)) such that at each point $(x, y, u) \in S$ we have that

$$(a(x,y), b(x,y), c(x,y,u)) \in T_{(x,y,u)}S\,,$$

where $c(x,y,u) := c_0(x,y)u(x,y) + c_1(x,y)$.

In order to do this, for all s, we consider the curve $(x(t,s), y(t,s), \tilde{u}(t,s))$ given by solving the following system of ODEs

$$\begin{cases} \dfrac{\mathrm{d}x(t,s)}{\mathrm{d}t} = a(x(t,s), y(t,s)) \\ \dfrac{\mathrm{d}y(t,s)}{\mathrm{d}t} = b(x(t,s), y(t,s)) \\ \dfrac{\mathrm{d}\tilde{u}(t,s)}{\mathrm{d}t} = c_0(x(t,s), y(t,s))\tilde{u}(t,s) + c_1(x(t,s), y(t,s))\,, \end{cases} \tag{2.3.2}$$

with initial conditions

$$\begin{cases} x(0,s) = x_0(s) \\ y(0,s) = y_0(s) \\ \tilde{u}(0,s) = \tilde{u}_0(s)\,, \end{cases}$$

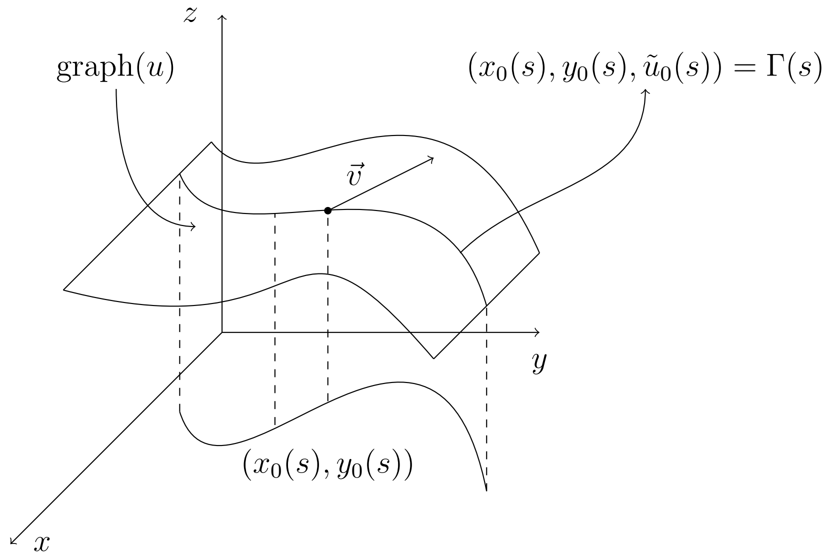

Figure 2.7: The initial curve Γ and the construction of the solution surface.

i.e., we require the initial point to be $\Gamma(s)$. The solutions to this system of ODEs (as one varies s) are the *characteristic equations* associated to the PDE (2.3.1) in consideration.

Note that, by definition, the characteristic curves $t \mapsto (x(t,s), y(t,s), \tilde{u}(t,s))$ have tangent vector

$$(a(x(t,s), y(t,s)), b(x(t,s), y(t,s)), c(x(t,s), y(t,s), \tilde{u}(t,s)))\,.$$

Hence we define the surface S as the union of this curve, namely

$$S := \{(x(t,s), y(t,s), \tilde{u}(t,s)) \in \mathbb{R}^3\}\,,$$

which is a parameterized representation of the solution surface graph(u) in the variables (t,s). Then we shall reexpress (whenever possible) the surface in terms of (x,y) as

$$u(x(t,s), y(t,s)) = \tilde{u}(t,s)\,.$$

Remark 2.3.1. If one does not want to think at the method of characteristics through the geometric interpretation above, one can think as follows. Let u be a solution of (2.3.1). Moreover let $t \mapsto (x(t), y(y))$ be a curve solving

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = a(x(t), y(t)) \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = b(x(t), y(t)) \end{cases}$$

and consider also the curve $t \mapsto u(x(t), y(t))$. Then, applying the chain rule, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[u(x(t), y(t))] = \dot{x}u_x + \dot{y}u_y = au_x + bu_y = c(x(t), y(t), u(x(t), y(t)))\,.$$

In other words, u along the curve $(x(t), y(t))$ coincides with the solution of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{u}(t) = c(x(t), y(t), \tilde{u}(t))\,,$$

provided of course that they start from the same initial condition.

Example 2.3.2. Consider the following Cauchy problem

$$\begin{cases} u_x + u_y = 1 \\ u(x, 0) = 2x^3\,. \end{cases}$$

We parameterize the initial condition with the curve $\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s)) = (s, 0, 2s^3)$.

In the notation of (2.3.1), here we have $a = 1$, $b = 1$, $c_0 = 0$ and $c_1 = 1$. Hence, following the procedure described above, we obtain the ODE system

$$\begin{cases} \dfrac{\mathrm{d}x(t,s)}{\mathrm{d}t} = a(x(t,s), y(t,s)) = 1 \\ \dfrac{\mathrm{d}y(t,s)}{\mathrm{d}t} = b(x(t,s), y(t,s)) = 1 \\ \dfrac{\mathrm{d}\tilde{u}(t,s)}{\mathrm{d}t} = c_0(x(t,s), y(t,s))\tilde{u}(t,s) + c_1(x(t,s), y(t,s)) = 1\,, \end{cases}$$

together with initial conditions

$$\begin{cases} x(0,s) = x_0(s) = s \\ y(0,s) = y_0(s) = 0 \\ \tilde{u}(0,s) = \tilde{u}_0(s) = 2s^3\,. \end{cases}$$

Therefore the characteristic curves are given by

$$\begin{cases} x(t,s) = s + t \\ y(t,s) = t \\ \tilde{u}(t,s) = 2s^3 + t\,. \end{cases}$$

Since we are looking for a solution in (x, y) coordinates, we have to find the inverse map $(t, s) \mapsto (x, y)$ to find $u(x, y)$. In this case it is very easy, indeed

$$\begin{cases} x(t,s) = s + t \\ y(t,s) = t \end{cases} \implies \begin{cases} t = y \\ s = x - t = x - y\,. \end{cases}$$

Hence the solution to the PDE is given by

$$u(x,y) = \tilde{u}(t(x,y), s(x,y)) = \tilde{u}(y, x-y) = 2(x-y)^3 + y\,.$$

Remark 2.3.3. There is no unique way to parameterize the initial condition. We could have define $\Gamma(s) = (s^4, 0, 2s^{12})$, which gives the same initial conditions as before. The parameterized solution surface is then given by the relation

$$u(x(t,s), y(t,s)) = \tilde{u}(t,s) = 2s^{12} + t\,.$$

Summary

- First order PDEs relate the solution surface to its tangent plane.
- They can be solved using the method of characteristics.
- The initial curve is a parametrization of the initial conditions and it is used to obtain the characteristic equations.
- The characteristic curves span the solution surface.

2.4. Existence and uniqueness questions

We now discuss some conditions that guarantee local existence and uniqueness. The questions is: under which conditions does there exist a unique integral surface for (2.2.1) that contains the initial curve Γ?

To solve our Cauchy problem, we need to solve the characteristic equations using the points we selected on Γ as an initial condition for the system of ODEs (2.3.2). Assuming that the coefficients of the ODEs are smooth (a and b have to be C^1), we can apply the Cauchy–Lipschitz theorem for ODEs that guarantees local existence in time and uniqueness of the solution. Hence, for all $s \in \mathbb{R}$ there exists some time interval $I_s = (s-\varepsilon, s+\varepsilon) \subseteq \mathbb{R}$ such that the solution $t \mapsto (x(t,s), y(t,s), \tilde{u}(t,s))$ exists uniquely for all $t \in I_s$.

Once solved the ODE system, we have an expression for $\tilde{u}$ in the variables (t,s). The fundamental relation between $\tilde{u}(t,s)$ and $u(x,y)$ (the desired solution) is given by $\tilde{u}(t,s) = u(x(t,s), y(t,s))$. Some difficulties may arise in the inversion of the transformation from (t,s) to (x,y), because the mapping $x = x(t,s)$, $y = y(t,s)$ may be not invertible. Thanks to the implicit function theorem we know that this map is locally invertible if the following *transversality condition* holds

$$\det \begin{pmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{pmatrix} \neq 0\,.$$

Note that this condition at $(0,s)$ becomes

$$\det \begin{pmatrix} a(x_0(s), y_0(s), u_0(s)) & b(x_0(s), y_0(s), u_0(s)) \\ \dfrac{\mathrm{d}}{\mathrm{d}s} x_0(s) & \dfrac{\mathrm{d}}{\mathrm{d}s} y_0(s) \end{pmatrix} \neq 0\,,$$

because $x(0,s) = x_0(s)$, $y(0,s) = y_0(s)$, $\frac{\partial x}{\partial t} = a$ and $\frac{\partial y}{\partial t} = b$.

Asking that the transversality condition is verified means that the vectors (a,b) and $(\frac{\mathrm{d}}{\mathrm{d}s} x_0(s), \frac{\mathrm{d}}{\mathrm{d}s} y_0(s))$ are transverse, i.e., they are not parallel, see Figure 2.8. Since

$$(a(x_0(s), y_0(s), u_0(s)), b(x_0(s), y_0(s), u_0(s)))$$

is the tangent vector to the characteristic $t \mapsto (x(t,s), y(t,s))$ at $t = 0$, while $(\frac{\mathrm{d}}{\mathrm{d}s} x_0(s), \frac{\mathrm{d}}{\mathrm{d}s} y_0(s))$ is the tangent vector to $\pi(\Gamma) := \{(x_0(s), y_0(s), 0) \in \mathbb{R}^3\}$ at s, the transversality condition means that $\pi(\Gamma)$ and $t \mapsto (x(t,s), y(t,s))$ are transverse at $t = 0$.

So far we discussed local problems (i.e., problems forbidding local existence of a solution), but we can also encounter obstacles to global existence. Indeed global existence (i.e., existence of a solution in all of the domain of interest) can fail for several reasons:

(i) In general ODEs only have local solutions and solutions can blow up in finite time. Similarly, solutions of Cauchy problem can blow up if you move far enough away from Γ.

(ii) If the characteristics $t \mapsto (x(t,s), y(t,s))$ intersect the Cauchy curve Γ more than once, then global existence may fail. This is because the characteristic equation is well-posed for a single initial condition. Think about the fact that a characteristic curve "carries" with it a charge of information from its intersection point with Γ. If a characteristic curve intersects Γ more than once these two "information charges" might be in conflict (see Figure 2.9).

(iii) If the vector field (a,b) vanishes at some point, then the corresponding PDE may only have a solution outside of a neighborhood of this point.

(iv) If the characteristics intersect within the domain of interest, then existence can break down at the intersection points.

Alternatively it may happen that the characteristic curves (as curves in $\mathbb{R}^3$) project all onto the same curve in the (x, y)-plane. In this case:

(i) either the characteristic curves coincide before the projection and thus there are infinitely many solutions;

(ii) or these curves do not coincide, which means that graph(u) should take different values on $\pi(\Gamma)$, which is impossible.

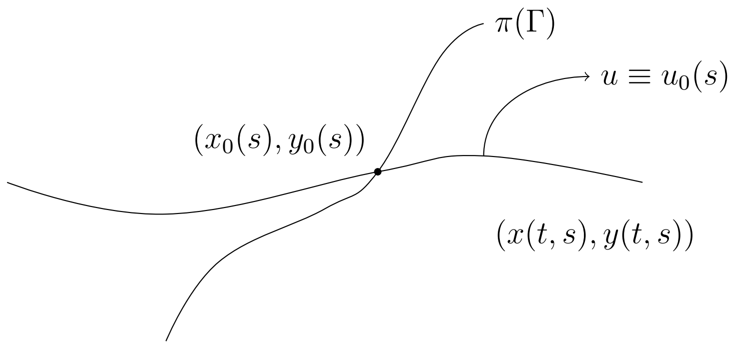

Figure 2.8: Projection of characteristic curve crossing $\pi(\Gamma)$ transversally.

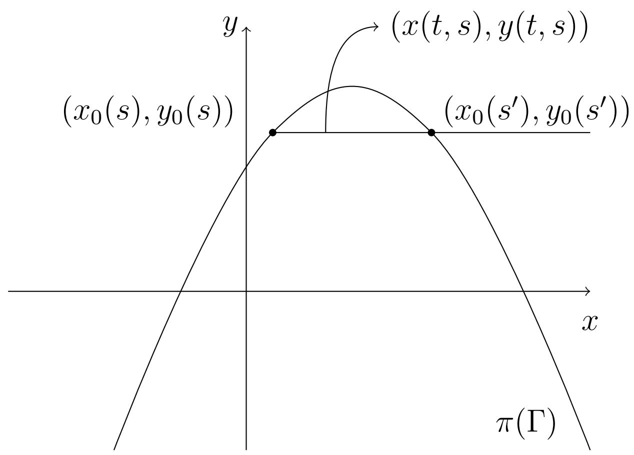

Figure 2.9: Projection of a characteristic curve crossing $\pi(\Gamma)$ twice. The value of u at $(x_0(s'), y_0(s'))$ may not be uniquely defined since it should both be equal to $u_0(s')$ and to $u(x(t', s'), y(t', s'))$.

2.5. Examples of existence and uniqueness

Later we will see the exact statement of the existence theorem. Before that, let us see some other examples.

Example 2.5.1. Consider the following Cauchy problem

$$\begin{cases} u_x + u_y = 1 \\ u(x, -x) = g(x)\,, \end{cases}$$

where g is any function in one variable. We parameterize the initial curve by choosing

$$\Gamma(s) := (s, -s, g(s))\,.$$

The characteristic equations are then given by

$$\begin{cases} \dfrac{\mathrm{d}x(t,s)}{\mathrm{d}t} = 1,\ x(0,s) = s & \Longrightarrow\ x(t,s) = t + s \\ \dfrac{\mathrm{d}y(t,s)}{\mathrm{d}t} = 1,\ y(0,s) = -s & \Longrightarrow\ y(t,s) = t - s \\ \dfrac{\mathrm{d}\tilde{u}(t,s)}{\mathrm{d}t} = 1,\ \tilde{u}(s,0) = g(s) & \Longrightarrow\ \tilde{u}(t,s) = g(s) + t\,. \end{cases}$$

In this case the relation between the variables x and y and the variables t and s is just

$$\begin{cases} t = \frac{x+y}{2} \\ s = \frac{x-y}{2}\,. \end{cases}$$

Therefore $\tilde{u}(t,s) = u(x(t,s), y(t,s))$ gives us the solution $u(x,y) = g\left(\frac{x-y}{2}\right) + \frac{x+y}{2}$.

Example 2.5.2. Let us now consider the same PDE as in the previous example, but with a different initial condition, namely

$$\begin{cases} u_x + u_y = 1 \\ u(x, x) = h(x)\,, \end{cases}$$

for a function h in one variable. As a parametrization of the initial curve we choose $\Gamma(s) := (s, s, h(s))$. Hence we have the following characteristic equations

$$\begin{cases} \dfrac{\mathrm{d}x(t,s)}{\mathrm{d}t} = 1,\ x(0,s) = s & \Longrightarrow\ x(t,s) = t + s \\ \dfrac{\mathrm{d}y(t,s)}{\mathrm{d}t} = 1,\ y(0,s) = s & \Longrightarrow\ y(t,s) = t + s \\ \dfrac{\mathrm{d}\tilde{u}(t,s)}{\mathrm{d}t} = 1,\ \tilde{u}(0,s) = h(s) & \Longrightarrow\ \tilde{u}(t,s) = h(s) + t\,. \end{cases}$$

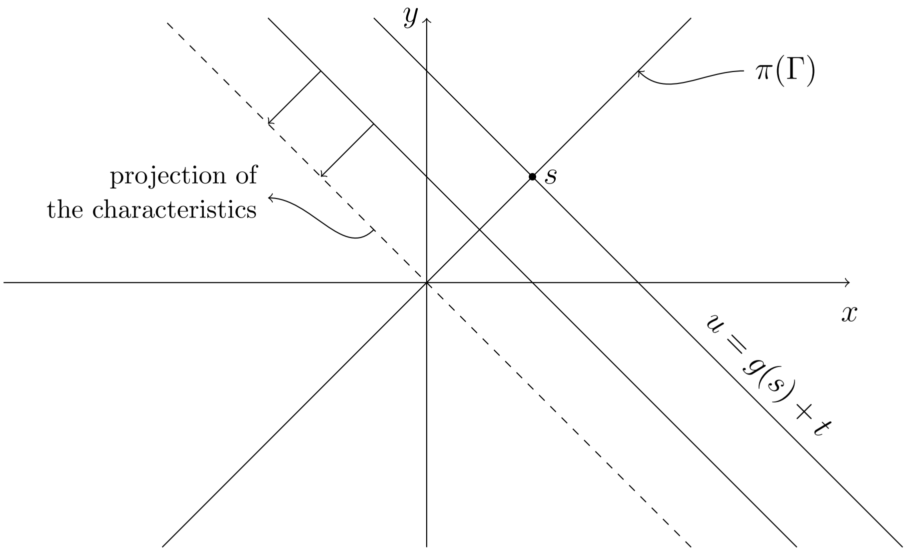

Figure 2.10: A plot of the characteristics in Example 2.5.1.

In this case the relation between (t, s) and $(x(t, s), y(t, s))$ cannot be inverted. This can be seen evaluating the determinant

$$\det \begin{pmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0\,.$$

Note that the projection of the initial curve is the diagonal $\{x = y\}$, but this is also the projection of a characteristic curve. In this case where $h(x) = x + c$ for a constant $c \in \mathbb{R}$, we obtain $\tilde{u}(t, s) = s + t + c$. Then it is not necessary to invert the mapping $(x(t, s), y(t, s))$ because $u = \frac{x+y}{2} + c + f\left(\frac{x-y}{2}\right)$ is a solution for every differentiable function f that vanishes at the origin. However, for any other choice of h the problem has no solutions.

Example 2.5.3. Consider the following Cauchy problem

$$\begin{cases} 2u_x + u_y = 1 - u \\ u(x, e^x + \frac{x}{2}) = x, \quad \text{for } x \in \mathbb{R}\,. \end{cases}$$

Hence we can choose

$$\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s)) = (s, e^s + \frac{s}{2}, s)$$

and the characteristic equations are

$$\begin{cases} \dfrac{\mathrm{d}x(t,s)}{\mathrm{d}t} = 2\,,\ x(0,s) = s & \implies x(t,s) = 2t + s \\ \dfrac{\mathrm{d}y(t,s)}{\mathrm{d}t} = 1\,,\ y(0,s) = e^s + \dfrac{s}{2} & \implies y(t,s) = e^s + \dfrac{s}{2} + t \\ \dfrac{\mathrm{d}\tilde{u}(t,s)}{\mathrm{d}t} = 1 - \tilde{u}(t,s)\,,\ \tilde{u}(0,s) = s\,. \end{cases}$$

Let us solve the equation for $\tilde{u}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{u} + \tilde{u} = 1 \implies \left(\frac{\mathrm{d}}{\mathrm{d}t}\tilde{u} + \tilde{u}\right) e^t = e^t \implies \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}e^t) = e^t\,.$$

Therefore we have

$$\tilde{u}(t,s)e^t - \tilde{u}(0,s) = \int_0^t e^\tau \,\mathrm{d}\tau = e^t - 1$$

$$\implies \tilde{u}(t,s) = e^{-t}[\tilde{u}(0,s) + e^t - 1] = e^{-t}s + 1 - e^{-t}\,. \tag{2.5.1}$$

From the relations $x = 2t+s$ and $2y = s+2e^s+2t$, we get $2y = 2e^s+x$. Therefore, since $e^s > 0$, it must hold $2y > x$ and $s = \ln(y - \frac{x}{2})$. Going back to the relation $x = 2t + s$, we then get $t = \frac{x-s}{2} = \frac{1}{2}(x - \ln(y - \frac{x}{2}))$. Plugging these relations in (2.5.1) and using that $u(x,y) = \tilde{u}(t(x,y), s(x,y))$, we obtain

$$u(x,y) = e^{-\frac{1}{2}(x-\ln(y-\frac{x}{2}))}[\ln(y - \frac{x}{2}) - 1] + 1 = e^{-\frac{1}{2}x}\sqrt{y - \frac{x}{2}}[\ln(y - \frac{x}{2}) - 1] + 1\,.$$

Remark 2.5.4. Computing the Jacobian for the initial curve we get

$$\det \begin{pmatrix} 2 & 1 \\ 1 & e^s + \frac{1}{2} \end{pmatrix} = 2e^s > 0 \quad \text{for all } s \in \mathbb{R}\,.$$

Therefore, we expect local existence of a unique solution at each point where $s = \ln(y - \frac{x}{2})$ is well defined, that is in the half plane $\{(x,y) : 2y > x\}$.

Example 2.5.5. We refer to [Pin05, Example 2.6]. Consider the PDE

$$\begin{cases} -yu_x + xu_y = u \\ u(x,0) = \psi(x)\,. \end{cases}$$

We parameterize the initial curve as

$$\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s)) = (s, 0, \psi(s))\,,$$

so the system of ODEs is

$$\begin{cases} \dfrac{\mathrm{d}x(t,s)}{\mathrm{d}t} = -y\,, \ x(0,s) = s \\ \dfrac{\mathrm{d}y(t,s)}{\mathrm{d}t} = x\,, \ y(0,s) = 0 \\ \dfrac{\mathrm{d}\tilde{u}(t,s)}{\mathrm{d}t} = \tilde{u}(t,s)\,, \ \tilde{u}(0,s) = \psi(s)\,. \end{cases}$$

Note that

$$\frac{\mathrm{d}^2}{\mathrm{d}^2 t}x = -\frac{\mathrm{d}}{\mathrm{d}t}y = -x \qquad \text{and} \qquad \frac{\mathrm{d}^2}{\mathrm{d}^2 t}y = \frac{\mathrm{d}}{\mathrm{d}t}x = -y\,.$$

Hence we obtain that

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}^2 t}x = -x & \Longrightarrow \ x(t,s) = f_1(s)\cos(t) + f_2(s)\sin(t) \\ \dfrac{\mathrm{d}^2}{\mathrm{d}^2 t}y = -y & \Longrightarrow \ y(t,s) = g_1(s)\cos(t) + g_2(s)\sin(t) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{u} = \tilde{u} & \Longrightarrow \ \tilde{u}(t,s) = \tilde{u}(0,s)e^t = \psi(s)e^t\,. \end{cases}$$

Using that $x(0,s) = s$, $y(0,s) = 0$, $\frac{\mathrm{d}}{\mathrm{d}t}x(0,s) = -y(0,s)$, $\frac{\mathrm{d}}{\mathrm{d}t}y(0,s) = x(0,s) = s$ and $\tilde{u}(0,s) = \psi(s)$, we obtain that

$$\begin{cases} x(t,s) = s\cos(t) \\ y(t,s) = s\sin(t)\,. \end{cases}$$

If we assume $s > 0$, we note that s, t act as polar coordinates, so we can invert the relation above to obtain

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ t = \arctan(y/x)\,. \end{cases}$$

This gives us the solution

$$u(x,y) = \psi(\sqrt{x^2+y^2})e^{\arctan(y/x)}\,.$$

2.6. The existence and uniqueness theorem

As we have seen in the previous section, existence and uniqueness of solutions is a delicate issue. In fact:

(i) The projection of the characteristics may not be transversal to the initial curve, in which case we are not able to express t and s in terms of x and y.

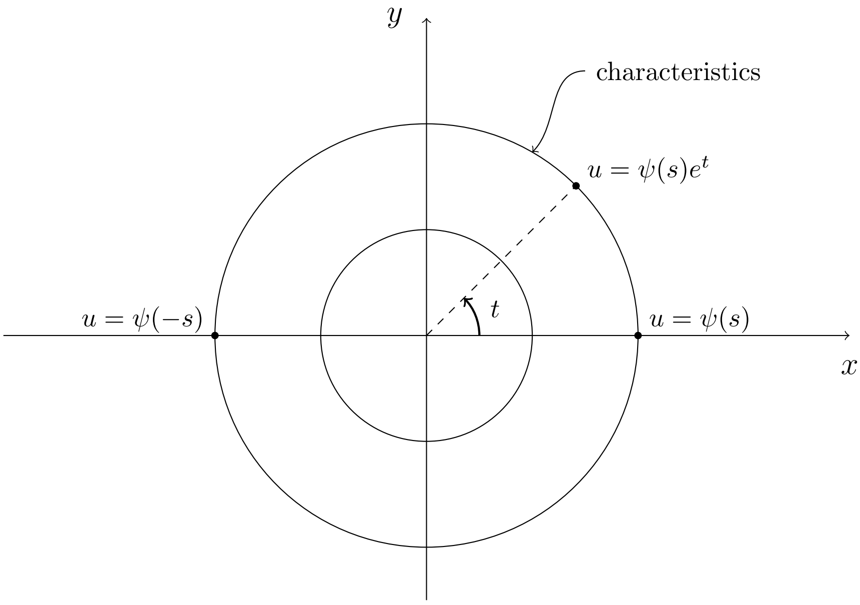

Figure 2.11: A plot of the characteristics in Example 2.5.5.

(ii) The projection of a characteristic may intersect $\pi(\Gamma)$ more than once, in which case the value of u may not be uniquely determined.

We have the following local existence and uniqueness result.

Theorem 2.6.1. *Consider the Cauchy problem*

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \\ \Gamma(s) = (x_0(s), y_0(s), u_0(s)) \,. \end{cases}$$

Assume that there exists $s_0 \in \mathbb{R}$ such that the transversality condition holds at $(0, s_0)$, i.e.,

$$\det \begin{pmatrix} \dfrac{\partial x}{\partial t}(0,s_0) & \dfrac{\partial y}{\partial t}(0,s_0) \\ \dfrac{\partial x}{\partial s}(0,s_0) & \dfrac{\partial y}{\partial s}(0,s_0) \end{pmatrix} \neq 0 \,.$$

Then there exists a unique solution u of the Cauchy problem defined in a neighborhood of $(x_0(s_0), y_0(s_0))$.

Sketch of the proof. We first solve the characteristic equations for s close to s_0. By existence and uniqueness for ODEs, we know that there exists a unique solution $(x(t,s), y(t,s), \tilde{u}(t,s))$ defined for (t,s) close to $(0,s_0)$. Thanks to the transversality condition, we know that we can apply the implicit function theorem and the map $(t,s) \mapsto (x(t,s), y(t,s))$ is invertible close to $(0, s_0)$. So this allows us to find a formula for u as $u(x,y) = \tilde{u}(x(t,s), y(t,s))$ in a neighborhood of $(x(0,s_0), y(0,s_0)) = (x_0(s_0), y_0(s_0))$. $\square$

Summary

- The method of characteristics can be used to solve first order quasilinear PDEs.
- The initial condition is described by the initial curve.
- The characteristics are expressed in terms of (t,s) instead of (x,y).
- We need to express the solution for u in terms of (x,y) but the map $(t,s) \mapsto (x,y)$ may be not invertible.

CHAPTER 3

CONSERVATION LAWS AND SHOCK WAVES

In this chapter we study an important class of first order PDEs called *conservation laws*, which are PDEs that prescribe conserved quantities such as mass, electric charge, number of cars (in traffic dynamics), number of people (in crowd dynamics), etc. We see some examples (as Burgers' equation with various initial data) and how we can apply the method of characteristics to solve conservation laws. However, solutions of conservation laws may develop discontinuities even for smooth initial data, for which reason we need to introduce the notion of weak solution.

We recall that the local existence theorem for first order quasilinear PDEs states that, under suitable conditions, one can find local solutions to first order quasilinear PDEs using the method of characteristics. We see in some examples that, even if a classical solution ceases to exist, the phenomenon (say for example the traffic flow) that we are modelling certainly does not. Therefore we broaden our definition of solution to allow us to make predictions about the phenomenon under study after the time when a classical solution ceases to exist.

3.1. What are (scalar) conservation laws?

Conservation laws are PDEs describing the evolution of a conserved quantity.

Definition 3.1.1. A *scalar conservation law* for a function $u\colon \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ of one spatial variable $x \in \mathbb{R}$ and one time variable $y \in [0,+\infty)$ is a PDE of the form

$$u_y + f(u)_x = 0\,, \tag{3.1.1}$$

for a given function $f\colon \mathbb{R} \to \mathbb{R}$, called *the flux function*. Equivalently, (3.1.1) can be written as

$$u_y + c(u)u_x = 0\,, \tag{3.1.2}$$

where $c(u) = f'(u)$.

Example 3.1.2. The easiest example of conservation law is the transport equation

$$u_y + cu_x = 0\,, \tag{3.1.3}$$

for a constant $c \in \mathbb{R}$, i.e., $c(u) = c$ in this case. Note that $u_y + cu_x = u_y + (cu)_x$, thus the flux is $f(u) = cu$ and $f'(u) = c(u) = c$.

If $u \geq 0$, u can represent the concentration of a pollutant in a river at time y and position x (see also Example 1.3.1). The constant $c \in \mathbb{R}$ represents the velocity of the river: if $c > 0$ the flow is from the left to right, if $c < 0$ the flow goes from right to left. Moreover, the total amount of pollutant in an interval $[a, b]$ at time y is

$$\int_a^b u(x, y)\,\mathrm{d}x\,.$$

The initial value problem (or Cauchy problem) for equation (3.1.3) is

$$\begin{cases} u_y + cu_x = 0\,, & (x, y) \in \mathbb{R} \times (0, \infty) \\ u(x, 0) = g(x)\,, & x \in \mathbb{R}\,, \end{cases}$$

where $g > 0$ is the concentration of the pollutant at time 0.

Let us solve this problem using the method of characteristics. Note that (3.1.3) is in the form (2.3.1). We can choose as initial curve $\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s)) = (s, 0, g(s))$. Hence the characteristic ODEs are

$$\begin{cases} \dfrac{\mathrm{d}x(t,s)}{\mathrm{d}t} = c,\ \ x(0, s) = s & \Longrightarrow\ x(t, s) = ct + s \\ \dfrac{\mathrm{d}y(t,s)}{\mathrm{d}t} = 1,\ \ y(0, s) = 0 & \Longrightarrow\ y(t, s) = t \\ \dfrac{\mathrm{d}\tilde{u}(t,s)}{\mathrm{d}t} = 0,\ \ \tilde{u}(0, s) = g(s) & \Longrightarrow\ \tilde{u}(t, s) = g(s)\,. \end{cases}$$

Now we need to invert the function $(t, s) \mapsto (x(t, s), y(t, s))$, which we can do and we obtain

$$\begin{cases} y(t, s) = t \\ x(t, s) = ct + s = cy + s \end{cases} \Longrightarrow \begin{cases} t(x, y) = y \\ s(x, y) = x - cy\,. \end{cases}$$

As a result, we get

$$u(x, y) = \tilde{u}(t(x, y), s(x, y)) = g(s(x, y)) = g(x - cy)\,,$$

which is a so-called *traveling wave*, se Figure 3.1.

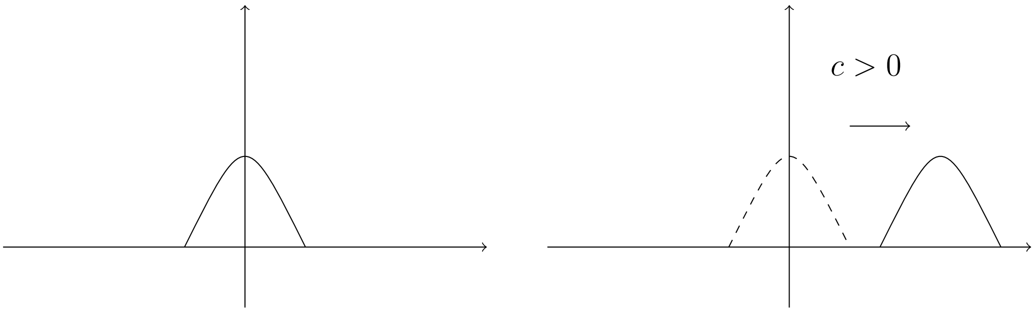

Figure 3.1: A travelling wave. On the left the initial condition $g(x)$ at $y = 0$. On the right, the solution $g(x - cy)$ when $c > 0$ after time $y > 0$.

3.2. Example: Burgers' equation

Example 3.2.1. Let us consider the Burgers' equation

$$\begin{cases} u_y + uu_x = 0 \\ u(x,0) = h(x)\,, \end{cases} \tag{3.2.1}$$

which models the flow of a mass with concentration $u(x, y)$, where the speed of the flow depends on the concentration. The variable y has the physical interpretation of a time and $h(x)$ is the initial condition, so the concentration of mass at time $y = 0$.

Remark 3.2.2. Burgers' equation is in the form $u_y + c(u)u_x = 0$ with $c(u) = u$, thus it is equivalent to the equation

$$u_y + \left(\frac{1}{2}u^2\right)_x = 0\,.$$

In particular the flux is $f(u) = u^2/2$ and the wave speed is $c(u) = f'(u) = u$.

Since (3.2.1) is a first order equation, we can use the method of characteristics. The parameterized initial condition is $\Gamma(s) = (s, 0, h(s))$ and the characteristic equations are given by

$$\begin{cases} \dfrac{\mathrm{d}x(t,s)}{\mathrm{d}t} = \tilde{u}(t,s)\,,\ x(0,s) = s & \Longrightarrow\ x(t,s) = s + h(s)t \\ \dfrac{\mathrm{d}y(t,s)}{\mathrm{d}t} = 1\,,\ y(0,s) = 0 & \Longrightarrow\ y(t,s) = t \\ \dfrac{\mathrm{d}\tilde{u}(t,s)}{\mathrm{d}t} = 0\,,\ \tilde{u}(0,s) = h(s) & \Longrightarrow\ \tilde{u}(t,s) = h(s)\,. \end{cases}$$

Inverting the function $(t,s) \mapsto (x(t,s), y(t,s)) = (s+h(s)t, t)$ as in the example of the transport equation is not possible, we just obtain that $y = t$ and $x = s+h(s)y$. Therefore the solution of the PDE is implicitly given by

$$u(s+h(s)y, y) = \tilde{u}(t,s) = h(s)\,.$$

Note that the initial value of u (namely h) determines the slope of the characteristic equations.

Now, recalling that $x = s+h(s)y$ and $\tilde{u}(t,s) = h(s)$, we have that $s = x - \tilde{u}y$. As a result, the solution can be written implicitly also as

$$u(x,y) = h(x-uy)\,.$$

Remark 3.2.3. This last implicit solution does not come unexpected (looking back at the solution of the transport equation) and it is actually a very general formula. Indeed, if you are solving a PDE in the form

$$\begin{cases} u_y + c(u)u_x = 0 & \text{for } (x,y) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}\,, \end{cases} s$$

then u satisfies the implicit equation

$$u(x,y) = u_0(x - c(u(x,y))y)\,.$$

Remark 3.2.4. Let us verify the transversality condition for (3.2.1) at a point $(0,s)$. We have that

$$\det \begin{pmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{pmatrix} = \det \begin{pmatrix} u & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0\,.$$

Therefore all the points of the initial curve Γ are not degenerate and, if h is continuously differentiable, Theorem 2.6.1 ensures that the conservation law has a unique solution on some time interval $[0, y_c)$ (the subscript c stands for "critical"), where $y_c > 0$ is sufficiently small.

Let us now determine the critical time y_c when the "classical" (or strong) solution breaks down.

Recall that $x = s + h(s)y$. Let us fix $y = \bar{y}$ and look at the map

$$s \mapsto s + h(s)\bar{y} = x(\bar{y}, s)\,.$$

Assuming that $h \in C^1$, we can compute

$$\frac{\mathrm{d}}{\mathrm{d}s}(s + h(s)\bar{y}) = 1 + h'(s)\bar{y}\,.$$

Assume also that, for all $s \in \mathbb{R}$, $1 + h'(s)\bar{y} > 0$. This implies that the map $s \mapsto x(\bar{y}, s)$ is strictly increasing, thus there exists its unique inverse map. Therefore, for $\bar{y}$ fixed, we can invert the relation $s \mapsto s + h(s)\bar{y}$ provided that $1 + h'(s)\bar{y} > 0$.

If we assume for instance that h' is globally bounded, then $1 + h'(s)\bar{y} > 0$ for $\bar{y} \geq 0$ small enough. What is the first value $\bar{y} > 0$ for which we cannot invert the relation $s \mapsto s + h(s)\bar{y}$? This is given by the first $\bar{y}$ for which there exists $s \in \mathbb{R}$ such that $1 + h'(s)y = 0$. If we denote by y_c such a $\bar{y}$, we can say that

$$y_c = \inf\left\{-\frac{1}{h'(s)} \ : \ s \in \mathbb{R},\ h'(s) < 0\right\} . \tag{3.2.2}$$

At time y_c, there is a problem with the solution u. To see it, we can differentiate the relation $u(s + h(s)y, y) = h(s)$ with respect to s to get

$$u_x(s + h(s)y, y)[1 + h'(s)y] = h'(s) .$$

Thus

$$u_x(s + h(s)y, y) = \frac{h'(s)}{1 + h'(s)y} ,$$

which shows that the derivative of u explodes when we take s and y_c such that $1+h'(s)y_c = 0$. Hence y_c is the critical time after which there is no smooth solution to the problem.

Remark 3.2.5. Note that the formula (3.2.2) is specific to Burgers' equation.

3.3. Critical time for conservation laws

Theorem 3.3.1. *Consider any scalar conservation law*

$$\begin{cases} u_y + c(u)u_x = 0 , & (x, y) \in \mathbb{R} \times (0, \infty) \\ u(x, 0) = u_0(x) , & x \in \mathbb{R} , \end{cases}$$

where $c, u_0 \in C^1(\mathbb{R})$ and $c \circ u_0 \colon \mathbb{R} \to \mathbb{R}$ is bounded with bounded derivative. Moreover define

$$\begin{aligned} y_c &:= \inf\left\{-\frac{1}{c(u_0(s))_s} \ : \ s \in \mathbb{R},\ c(u_0(s))_s < 0\right\} \\ &= \inf\left\{-\frac{1}{c'(u_0(s))u_0'(s)} \ : \ s \in \mathbb{R},\ c(u_0(s))_s < 0\right\} , \end{aligned}$$

with the standard convention that $y_c = \infty$ if $c(u_0(s))_s \geq 0$ for all $s \in \mathbb{R}$.

Then, if $y_c > 0$, there exists a unique solution to the PDE above in $[0, y_c)$ and u satisfies the implicit equation

$$u(x, y) = u_0(x - c(u(x, y))y) .$$

Remark 3.3.2. Solutions of conservation laws are constant along their characteristics, which are straight lines. Indeed, for each $s \in \mathbb{R}$, the characteristic through a point $(s, 0)$ is the line in the (x, y)-plane going through $(s, 0)$ with slope $1/c(u_0(s))$ and on this line u is equal to the constant $u_0(s)$.

Remark 3.3.3. If $c(u_0(s)) < 0$, then there exists a time when the characteristics cross. Heuristically you can think about the latter condition as when a faster characteristic starts from a point behind a slower characteristic. If $c(u_0(s))_s$ is never decreasing, there are not singularities, however such data are exceptional.

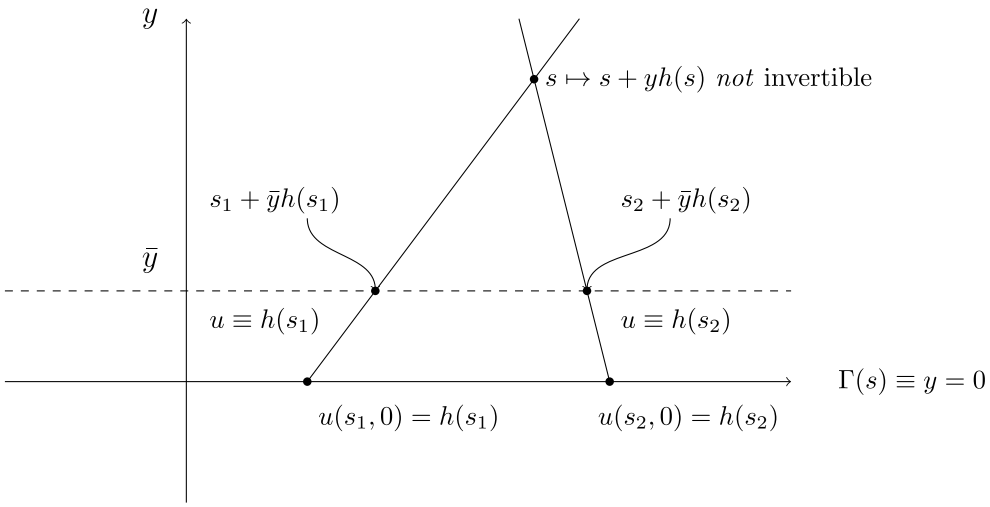

Figure 3.2: Crossing characteristics.

3.4. Notion of weak solutions

In the context of conservation laws, we say that u is a *classical* (or *strong*) solution if u satisfies the classical formulation of conservation laws (3.1.1), or equivalently (3.1.2). To introduce a notion of weak solution, we use the so-called *integral formulation* of conservation laws, that is

$$\int_a^b u(x, y_2)\,\mathrm{d}x - \int_a^b u(x, y_1)\,\mathrm{d}x = -\int_{y_1}^{y_2} [f(u(b, y)) - f(u(a, y))]\,\mathrm{d}y\,, \qquad (3.4.1)$$

for all $a < b$, $y_1 < y_2$, where $f(u)$ is the flux function.

The classical formulation (3.1.1) (or (3.1.2)) makes no sense if u is not continuously differentiable, but the integral formulation (3.4.1) still makes sense even if u is not continuous. Moreover we have that

- if u satisfies the classical formulation, then it satisfies also the integral formulation;
- if u satisfies the integral formulation and it is regular, then it is a classical solution.

Let us suppose that the domain of definition of u is D and D is divided into subdomains D_i for $i = 1, \ldots, n$. Assume that $u(x, y)$ is continuously differentiable in each D_i for all $i = 1, \ldots, n$.

Definition 3.4.1. We say that $u(x, y)$ is a *weak solution* on $D = \cup_{i=1}^n D_i$ if u satisfies the original PDE (3.1.1) (or (3.1.2)) in each D_i for $i = 1, \ldots, n$ and the integral form (3.4.1) on D. The boundaries between the regions D_i are curves called *shocks*.

Thus, in our notion of weak solution, we relax the requirement of a global classical solution and we allow solutions that are a combination of classical solutions on each D_i with possibly jumps between them. We now see a very important condition that has to be verified in order to have a global weak solution.

3.5. Rankine-Hugoniot condition

Given u as in Definition 3.4.1, let $x = \sigma(y)$ be a smooth curve in the (x, y)-plane across which u is discontinuous. Assume that u, u_x, u_y have one-sided limits as $x \to \sigma(y)^+$ and as $x \to \sigma(y)^-$. Choosing $a < \sigma(y) < b$, the formula (3.4.1) becomes

$$
\begin{aligned}
f(u(a,y)) - f(u(b,y)) &= \frac{\mathrm{d}}{\mathrm{d}y} \int_a^{\sigma(y)} u(x,y)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}y} \int_{\sigma(y)}^b u(x,y)\,\mathrm{d}x \\
&= \int_a^{\sigma(y)} u_y(x,y)\,\mathrm{d}x + \sigma'(y) u(\sigma(y)^-, y) + \int_{\sigma(y)}^b u_y(x,y)\,\mathrm{d}x - \sigma'(y) u(\sigma(y)^+, y)\,.
\end{aligned}
$$

In the limit as $a \to \sigma(y)^-$ and $b \to \sigma(y)^+$, the integral vanishes and we have

$$
f^+ - f^- = \sigma'(y)(u^+ - u^-) \qquad \Longrightarrow \qquad \sigma'(y) = \frac{f^+ - f^-}{u^+ - u^-}\,, \tag{RH}
$$

where $u^\pm = \lim_{x \to \sigma(y)^\pm} u(x, y)$ and $f^\pm = f(u^\pm)$. The condition (RH) is the *Rankine-Hugoniot condition* and, if a curve $x = \sigma(y)$ satisfies (RH), then we say that it is a *shock wave*.

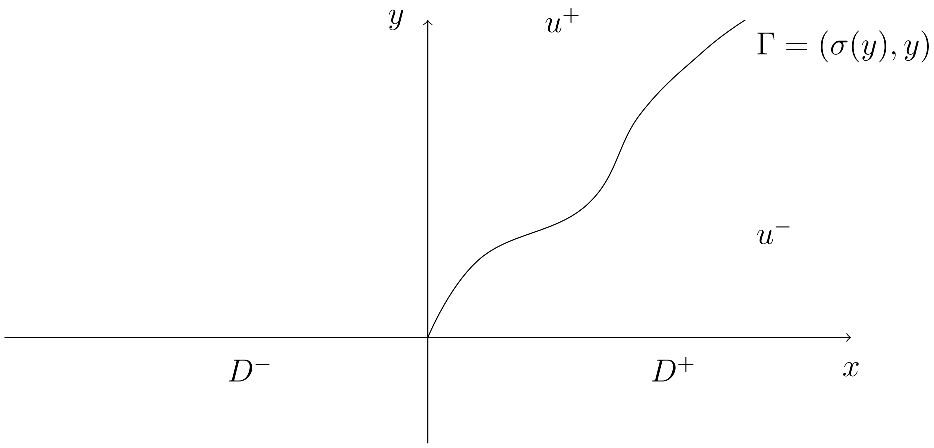

Figure 3.3: A solution with a jump discontinuity along a curve.

Example 3.5.1. Consider the following Burgers' equation

$$\begin{cases} u_y + uu_x = 0\,, & (x,y) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = e^{-x^2}\,, & x \in \mathbb{R}\,. \end{cases}$$

Note that in this case $c(u) = u$ and $u_0(x) = e^{-x^2}$. Observe that $c(u_0(s)) = e^{-s^2}$ is decreasing for $s > 0$. Hence this conservation law has a classical solution for $y \in [0, y_c)$ where

$$y_c = \min_{s>0} \left\{ -\frac{1}{c'(u_0(s))u_0'(s)} \right\} = \min_{s>0} \frac{e^{s^2}}{2s}\,.$$

Let $y_*(s) := e^{s^2}/2s$, then $\lim_{s\to 0} y_*(s) = \lim_{s\to+\infty} y_*(s) = +\infty$. Therefore y_* is minimized when

$$\frac{\mathrm{d}y_*}{\mathrm{d}s} = 0 \iff e^{s^2}\left(1 - \frac{1}{2s^2}\right) = 0 \iff s^2 = \frac{1}{2}\,.$$

Therefore $s = 1/\sqrt{2}$ is the unique critical point of y_* in $(0,\infty)$. We conclude that $y_c = y_*(1/\sqrt{2}) = e^{1/2}/\sqrt{2}$. Then, for $y < y_c$ the solution is

$$u(x,y) = u_0(s) - e^{-s^2}\,,$$

where s is the unique solution of $x = s + c(u_0(s))y = s + e^{s^2}y$.

3.6. The entropy condition

Example 3.6.1. We refer to [Pin05, Example 2.14]. Consider the Burgers' equation

$$\begin{cases} u_y + uu_x = 0 \\ u(x,0) = h(x)\,, \end{cases}$$

with initial condition h defined as

$$h(x) = \begin{cases} 1\,, & \text{if } x \le 0 \\ 1 - x/\alpha\,, & \text{if } 0 \le x \le \alpha \\ 0\,, & \text{if } \alpha \le x\,. \end{cases}$$

Since the initial condition $h(x)$ is not monotonously increasing, the solution develops a singularity at time $y = y_c$, where

$$y_c = \inf_{c(u_0(s))_s<0} \left\{ -\frac{1}{c'(u_0(s))u_0'(s)} \right\}.$$

Note that $c(u_0(s)) = c(h(s)) = h(s)$, however the initial datum $h(x)$ is not differentiable. Nevertheless $c(u_0(s)) = 1 - s/\alpha$ is decreasing for $s \in (0,\alpha)$ and we expect u to become discontinuous at time

$$y_c = \inf_{s\in(0,\alpha)} \left\{ -\frac{1}{c(u_0(s))_s} \right\} = \inf_{s\in(0,\alpha)} \{\alpha\} = \alpha\,.$$

The method of characteristic gives the system of ODEs

$$\begin{cases} x_t = a = \tilde{u} & \Longrightarrow\ x(t,s) = s + th(s) \\ y_t = b = 1 & \Longrightarrow\ y(t,s) = t \\ \tilde{u}_t = 0 & \Longrightarrow\ \tilde{u}(t,s) = h(s)\,. \end{cases}$$

Therefore, for $y < y_c = \alpha$, the solution is $u(x,y) = h(s)$, where (x,y) lies on the characteristic through $(s,0)$. Since h is defined piecewise, we have to consider three cases:

(i) If $s \le 0$, then $h(s) = 1$ and the characteristic have equation $x = s + h(s)y = s + y$. Therefore $u(x,y) = h(s) = 1$ for $s \le 0$, which holds if and only if $x \le y$ (see Figure 3.4).

(ii) If $s \ge \alpha$, then $h(s) = 0$ and the characteristics have equation $x = s + h(s)y = s$. Therefore $u(x,y) = h(s) = 0$ for $s \ge \alpha$, which holds if and only if $x \ge \alpha$ (see Figure 3.4).

(iii) Finally for $0 \leq s \leq \alpha$, we have $h(s) = 1 - s/\alpha$ and the characteristics have equation $x = s + h(s)y = s + (1 - s/\alpha)y$, which is equivalent to $s = \alpha(x-y)/(\alpha - y)$. Therefore

$$u(x,y) = h(s) = 1 - \frac{s}{\alpha} = 1 - \frac{1}{\alpha}\left(\frac{\alpha(x-y)}{\alpha - y}\right) = 1 - \frac{x-y}{\alpha - y} = \frac{\alpha - x}{\alpha - y}$$

if $0 \leq s \leq \alpha$, which holds if and only if $y \leq x \leq \alpha$.

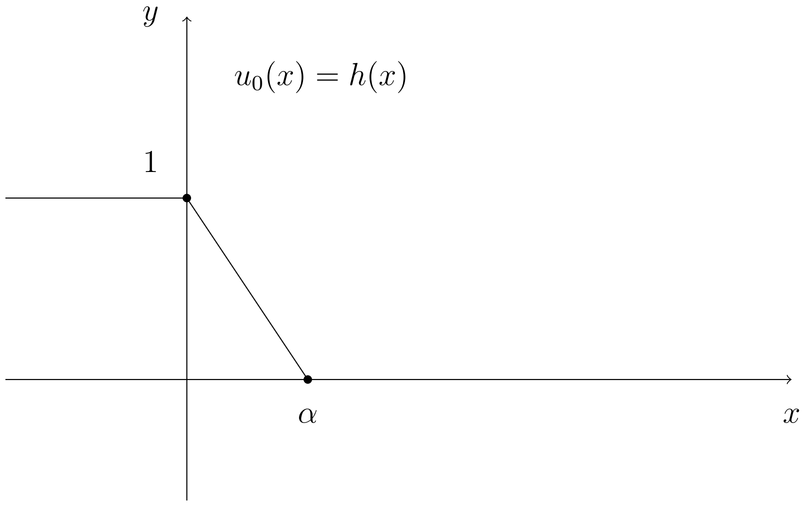

Figure 3.4: Initial condition of Example 3.6.1.

Note that at time $y_c = \alpha$ characteristics intersect. Indeed, from $y = y_c = \alpha$, a shock appears in the graph of u, jumping from 1 to 0. To find the shock curve $x = \gamma(y)$, we impose the Rankine-Hugoniot condition, so γ satisfies

$$\gamma'(y) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1}{2} \cdot \frac{(u^+)^2 - (u^-)^2}{u^+ - u^-} = \frac{u^+ + u^-}{2} = \frac{1}{2}\,.$$

Since $\gamma'(y) = 1/2$, we deduce that γ is a linear function and, by the condition that the shock starts at the point (α, α), we get

$$\gamma(y) = \alpha + \frac{1}{2}(y - \alpha)$$

for $y \geq \alpha$. Therefore, for $y \geq \alpha$, we can write the solution for u as

$$u(x,y) = \begin{cases} 1\,, & \text{if } x < \gamma(y) \\ 0\,, & \text{if } x > \gamma(y), \end{cases}$$

which is a weak solution (or shock wave). In conclusion, we constructed the following weak solution

$$u(x,y) = \begin{cases} 1\,, & \text{if } x \leq y,\ y \in [0,\alpha) \\ \dfrac{\alpha - x}{\alpha - y}\,, & \text{if } y \leq x \leq \alpha,\ y \in [0,\alpha) \\ 0\,, & \text{if } x \geq \alpha,\ y \in [0,\alpha) \\ 1\,, & \text{if } x < (y+\alpha)/2,\ y \in [\alpha,\infty) \\ 0\,, & \text{if } x > (y+\alpha)/2,\ y \in [\alpha,\infty)\,. \end{cases}$$

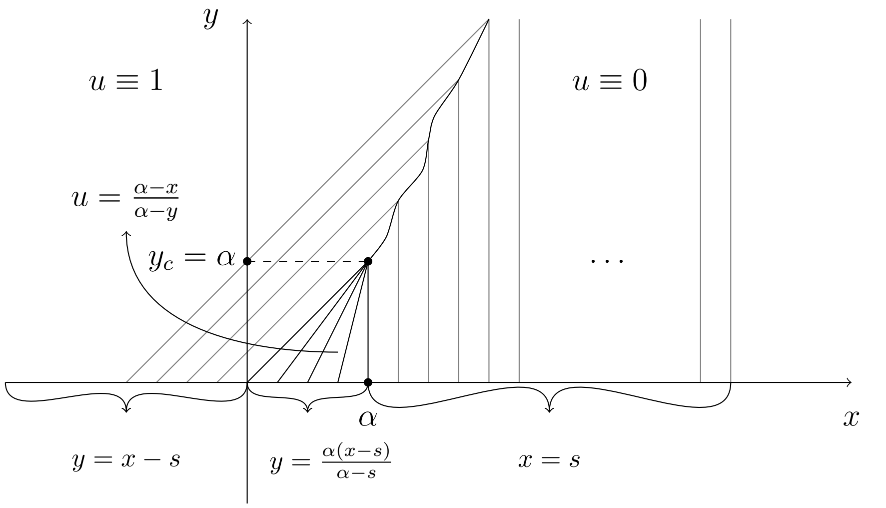

Figure 3.5: Intersection of characteristic curves in Example 3.6.1.

Example 3.6.2. We refer to [Pin05, Example 2.15]. In this example we see how, by allowing for weak solutions, we lose uniqueness. Consider the Burgers' equation

$$\begin{cases} u_y + uu_x = 0 \\ u(x,0) = h(x)\,, \end{cases}$$

with initial condition h defined as

$$h(x) = \begin{cases} 0\,, & \text{if } x \leq 0 \\ x/\alpha\,, & \text{if } 0 \leq x \leq \alpha \\ 1\,, & \text{if } x \geq \alpha\,. \end{cases}$$

Since $c(h(s))_s = h'(s) \geq 0$, there is no critical time $y_c > 0$, where the characteristics intersect. On the contrary, the characteristics diverge. In this situation we talk about *expansion waves*. The characteristic of x is given by

$$x(t,s) = s + h(s)y = \begin{cases} s\,, & \text{if } s \le 0 \\ s + y/\alpha\,, & \text{if } s \in [0,1] \\ s + y\,, & \text{if } s \ge \alpha\,. \end{cases}$$

Thus, inverting the relation in order to write s in terms of x, y, we have

$$s = \begin{cases} x\,, & \text{if } \{s \le 0\} = \{x \le 0\} \\ \dfrac{\alpha x}{\alpha + y}\,, & \text{if } \{s \le [0,\alpha]\} = \left\{0 \le \dfrac{\alpha x}{\alpha + y} \le \alpha\right\} = \{0 \le x \le y + \alpha\} \\ x - y\,, & \text{if } \{s \ge \alpha\} = \{x - y \ge \alpha\} = \{x \ge y + \alpha\}\,. \end{cases}$$

Since $u(x,y) = h(s)$, this yields

$$u(x,y) = \begin{cases} 0\,, & \text{if } x \le 0 \\ \dfrac{s}{\alpha} = \dfrac{x}{\alpha + y}\,, & \text{if } 0 \le x \le y + \alpha \\ 1\,, & \text{if } x \ge y + \alpha\,. \end{cases}$$

Let us now look at the case when $\alpha \to 0$. Then $h(x)$ becomes the step function

$$h(x) = \begin{cases} 0\,, & \text{if } x \le 0 \\ 1\,, & \text{if } x \ge 1\,. \end{cases}$$

By taking the solution u above and letting $\alpha \to 0$, we obtain

$$u(x,y) = \begin{cases} 0\,, & \text{if } x \le 0 \\ \dfrac{x}{y}\,, & \text{if } 0 < x < y \\ 1\,, & \text{if } x \ge y\,, \end{cases} \tag{3.6.1}$$

which is a classical solution.

In principle, we can still find a solution with a shock, because $u = 0$ for $x < 0$ and $u = 1$ for $x > y$. A shock should be a curve $(\gamma(y), y)$ satisfying $\gamma(0) = 0$ and $\gamma'(y) = (u^+ + u^-)/2 = 1/2$ by the Rankine-Hugoniot condition, which leads to $\gamma(y) = y/2$. Therefore another solution would be

$$u(x,y) = \begin{cases} 0\,, & \text{if } x \le y/2 \\ 1\,, & \text{if } x \ge y/2\,. \end{cases} \tag{3.6.2}$$

If a conservation law does not have a unique weak solution, then how can we select the "right" one? The answer comes from the following entropy condition.

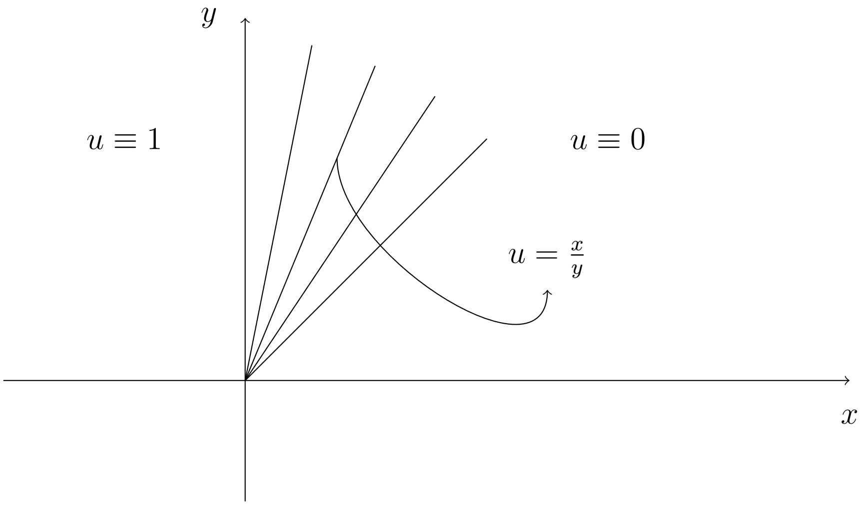

Figure 3.6: The expansion (or rarefaction) wave of Example 3.6.1.

Definition 3.6.3. A weak solution satisfies the *entropy condition* if characteristics only enter shock waves but do not emanate from them, i.e., a shock wave $x = \gamma(y)$ satisfies the entropy condition if $c(u^+) < \gamma' < c(u^-)$, or equivalently $f'(u^+) < \gamma' < f'(u^-)$, where f is the flux and $f'(u) = c(u)$.

One can see that the solution (3.6.1) satisfies the entropy condition trivially because it has no shocks, while (3.6.2) does not satisfy the entropy condition, since $c(u^+) = 1$, $c(u^-) = 0$ and $\gamma' = 1/2$.

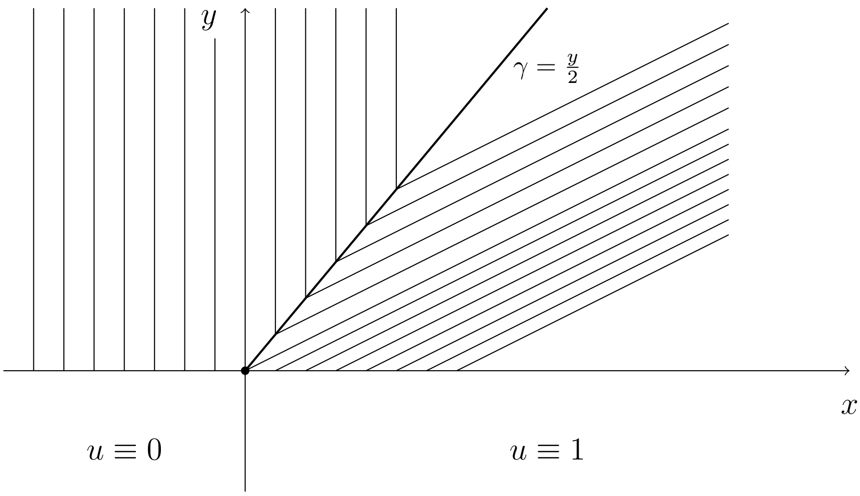

Figure 3.7: Characteristics emerging from the shock.

CHAPTER 4

ONE DIMENSIONAL WAVE EQUATION

In this section we study the one dimensional wave equation (which is the archetype of hyperbolic equation, see Section 6.1) on the real line. We use the reduction to the canonical form to show that the general solution of the one dimensional wave equation can be decomposed as superposition of a forward and a backward traveling wave. We also introduce the d'Alembert's formula that gives us an explicit solution to the Cauchy problem.

Usually real life applications of the wave equation take place on a finite interval of times. In that case, we would need to deal with boundary conditions but for now we consider the simplified setting in absence of boundary conditions, in order to make some general considerations.

4.1. Canonical form and general solution

The homogeneous one dimensional wave equation is a hyperbolic second order differential equation of the form

$$u_{tt} - c^2 u_{xx} = 0\,, \quad (x,t) \in \mathbb{R} \times (0,\infty)\,, \tag{4.1.1}$$

where $c \in \mathbb{R}$ represents the wave speed.

This is called wave equation because it well describes waves. Wave propagation appears in a huge plethora of different physical situations: water wave propagation, sound waves, seismic waves and light waves. It arises in acoustic, electromagnetism, and fluid dynamics.

Given (4.1.1), we introduce the new variables $\xi(x,t) = x + ct$ and $\eta(x,t) = x - ct$ and we set $w(\xi,t) = u(x(\xi,\eta), t(\xi,\eta))$. Using the chain rule on the function $u(x,t) = w(\xi(x,t), \eta(x,t))$, we obtain

$$\begin{aligned} u_t &= \frac{\partial}{\partial t}[w(\xi(x,t),\eta(x,t))] \\ &= w_\xi(\xi(x,t),\eta(x,t)) \cdot \xi_t(x,t) + w_\eta(\xi(x,t),\eta(x,t)) \cdot \eta_t(x,t) = w_\xi \xi_t + w_\eta \eta_t \end{aligned}$$

and

$$\begin{aligned}
u_x &= \frac{\partial}{\partial x}[w(\xi(x,t),\eta(x,t))] \\
&= w_\xi(\xi(x,t),\eta(x,t)) \cdot \xi_x(x,t) + w_\eta(\xi(x,t),\eta(x,t)) \cdot \eta_x(x,t) = w_\xi \xi_x + w_\eta \eta_x \,.
\end{aligned}$$

Since $\xi_x = \eta_x = 1$ and $\xi_t = c$, $\eta_t = -c$, we have

$$\begin{aligned}
u_t(x,t) &= c[w_\xi(x+ct, x-ct) - w_\eta(x+ct, x-ct)] \\
u_x(x,t) &= w_\xi(x+ct, x-ct) + w_\eta(x+ct, x-ct) \,.
\end{aligned}$$

Differentiating again with respect to x and t the above expressions, we get

$$\begin{aligned}
u_{tt} &= \frac{\partial}{\partial t}[c(w_\xi - w_\eta)] = c^2(w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta}) \\
u_{xx} &= \frac{\partial}{\partial x}[w_\xi + w_\eta] = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta} \,.
\end{aligned}$$

Plugging in the wave equation we obtain

$$0 = u_{tt} - c^2 u_{xx} = c^2[w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta} - w_{\xi\xi} - 2w_{\xi\eta} - w_{\eta\eta}] = -4c^2 w_{\xi\eta} \,.$$

Thus we have $w_{\xi\eta} = 0$. Note that $w_{\xi\eta} = \partial w_\xi / \partial \eta = 0$. This implies that w_ξ is independent of η, therefore we can write it as $w_\xi(\xi,\eta) = f(\xi)$, for some function $f\colon \mathbb{R} \to \mathbb{R}$. Then we integrate and we get

$$w(\xi,\eta) = \int_0^\xi f(\alpha)\,\mathrm{d}\alpha + G(\eta)$$

with $G(\eta) = w(0,\eta)$. If we define $F(\xi) = \int_0^\xi f(\alpha)\,\mathrm{d}\alpha$, we can write the general solution for the equation $w_{\xi\eta} = 0$ as follows

$$w(\xi,\eta) = F(\xi) + G(\eta) \,,$$

for $F, G \in C^2(\mathbb{R}) = \{f\colon \mathbb{R} \to \mathbb{R} \ : \ f' \text{ and } f'' \text{ exist and are continuous}\}$. Thus, in the original variables, the general solution of the one dimensional wave equation is

$$u(x,t) = F(x+ct) + G(x-ct) \,. \tag{4.1.2}$$

If u solves (4.1.1) then there exist $F, G \in C^2(\mathbb{R})$ such that (4.1.2) holds. Conversely any two functions $F, G \in C^2(\mathbb{R})$ give a solution of (4.1.1) via the formula (4.1.2).

Note that

- $G(x - ct)$ represents a wave moving to the right with velocity $c > 0$ and thus we say that it is a *forward wave*;

- $F(x+ct)$ is a wave moving to the left with velocity $c>0$, thus a *backward wave*.

Remark 4.1.1. Equation (4.1.2) shows that any solution of the one dimensional wave equation is the sum of two traveling waves.

Remark 4.1.2. Observe that the functions $F(x+ct)$ and $G(x-ct)$ are constant along lines of the form $x+ct=\alpha\in\mathbb{R}$ and $x-ct=\beta\in\mathbb{R}$, respectively. Those lines are called *characteristics*. Hence, for the wave equation the characteristics are straight lines in the (x,t)-plane with slopes $\pm 1/c$. As for first order PDEs, the "information" is propagated along these curves.

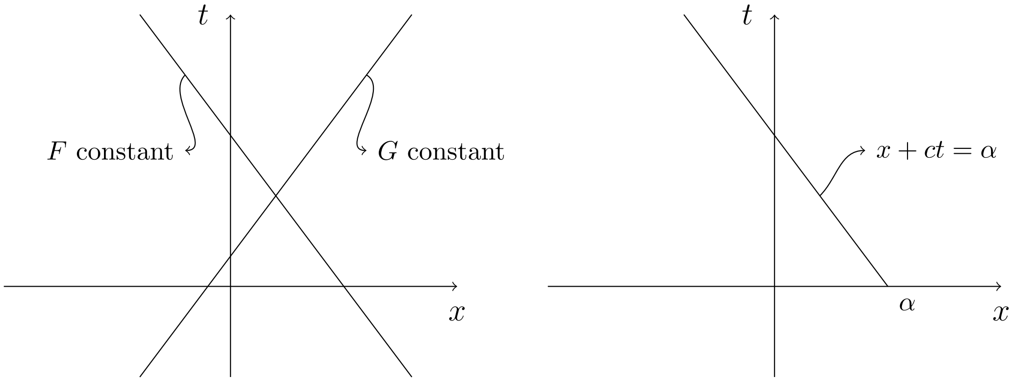

Figure 4.1: On the left, the characteristics where F and G are constant. On the right, the backward wave $F(x+ct)$.

We saw that (4.1.2) is valid for $F,G\in C^2(\mathbb{R})$. Let us now extend the validity of this equation. Consider F,G real piecewise continuous functions. Let us approximate F and G by two sequences of C^2 functions $\{F_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$, namely we demand that

(i) $F_n, G_n\in C^2$ for all $n\in\mathbb{N}$;

(ii) $F_n\to F$ at all continuity points of F;

(iii) $G_n\to G$ at all continuity points of G.

Then the function

$$u_n(x,t)=F_n(x+ct)+G_n(x-ct)$$

is a solution of (4.1.1) in the classical sense. Sending n to infinity, we obtain the function $u(x,t)=F(x+ct)+G(x-ct)$, which is not necessarily smooth enough to be a "classical" or "strong" solution, but we can say that u is a *generalized solution* of the wave equation.

Remark 4.1.3. Assume that u is a smooth function except at (x_0, t_0). Then either F is not smooth at $x_0 + ct_0$ or G is not smooth at $x_0 - ct_0$. Note that there are two characteristics passing through (x_0, t_0), which are $x - ct = x_0 - ct_0$ and $x + ct = x_0 + ct_0$. Thus, for any time $t_1 \neq t_0$, u is smooth except at one or two points $x_\pm$ that satisfy

$$x_- - ct_1 = x_0 - ct_0\,, \quad x_+ + ct_1 = x_0 + ct_0\,.$$

The singularities of solutions of the wave equation are traveling only along characteristics, which is a typical feature of hyperbolic equations.

4.2. The Cauchy problem and d'Alembert's formula

The Cauchy problem for the homogeneous one dimensional wave equation is given by

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0\,, & (x,t) \in \mathbb{R} \times (0, \infty) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x)\,, \end{cases} \tag{4.2.1}$$

where f and g represent respectively the amplitude and the velocity at time $t = 0$. A solution to the above Cauchy problem can be thought as the amplitude of the vibration of an infinite string. A classical solution for the Cauchy problem is a function u that is twice continuously differentiable for all $t \in \mathbb{R}^+$ and solving (4.2.1).

Since the general solution to (4.1.1) is given by (4.1.2), we need to find F and G using the initial conditions. By $u(x,0) = f(x)$, we deduce

$$f(x) = u(x,0) = F(x) + G(x)\,.$$

While, thanks to $u_t(x,0) = g(x)$, we get

$$g(x) = u_t(x,0) = \frac{\partial}{\partial t}[F(x+ct) + G(x-ct)]|_{t=0} = c[F'(x) - G'(x)]\,,$$

which implies that

$$F(x) - G(x) = \frac{1}{c}\int_0^x g(y)\,\mathrm{d}y + [F(0) - G(0)]\,.$$

Hence we have the system

$$\begin{cases} F(x) + G(x) = f(x) \\ F(x) - G(x) = \frac{1}{c}\int_0^x g(y)\,\mathrm{d}y + [F(0) - G(0)]\,. \end{cases}$$

Adding these two equations we get

$$2F(x) = f(x) + \frac{1}{c}\int_0^x g(y)\,\mathrm{d}y + [F(0) - G(0)]\,.$$

On the other hand, subtracting the second equation from the first equation, we have

$$2G(x) = f(x) - \frac{1}{c}\int_0^x g(y)\,\mathrm{d}y - [F(0) - G(0)]\,.$$

Therefore, the solution of (4.2.1) is given by

$$\begin{aligned} u(x,y) &= F(x+ct) + G(x-ct) \\ &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(y)\,\mathrm{d}y + \frac{F(0)-G(0)}{2} + \\ &\quad + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(y)\,\mathrm{d}y - \frac{F(0)-G(0)}{2}\,, \end{aligned}$$

which implies the so-called *d'Alembert's formula*

$$u(x,y) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y\,. \tag{4.2.2}$$

Remark 4.2.1. The value of the solution at (x,t) is only influenced by the values of f and g in $[x-ct, x+ct]$.

Remark 4.2.2. For the wave equation in higher dimension there are formulas similar to the d'Alembert's one, but they are more complicated and they go beyond the scope of these notes.

Example 4.2.3. Consider the Cauchy problem (4.2.1) with $c = 1$ and initial conditions given by

$$f(x) = \begin{cases} 0\,, & \text{if } |x| > 1 \\ 1 - x^2\,, & \text{if } |x| \le 1\,, \end{cases} \qquad g(x) = \begin{cases} 0\,, & \text{if } |x| > 1 \\ 1\,, & \text{if } -1 \le x \le 1\,. \end{cases}$$

Using d'Alembert's formula we can for example compute the solution u of (4.2.1) at the point $(1, 1/2)$ and we obtain that

$$u(1,1/2) = \frac{f(1+1/2)+f(1-1/2)}{2} + \frac{1}{2}\int_{1/2}^{3/2} g(y)\,\mathrm{d}y = \frac{3}{8} + \frac{1}{2}\int_{1/2}^{1}\mathrm{d}y = \frac{1}{2}\,.$$

Now observe that, since f is not C^1, u is also not C^1. However we claim that u is continuous, even if g is not continuous. Indeed, since f is continuous,

$(f(x+t)-f(x-t))/2$ is continuous as well, thus we just need to check that the second term in (4.2.2) is continuous. Given a sequence of points $(x_k,t_k)\to(x,t)$, we have

$$\begin{aligned}&\left|\frac{1}{2}\int_{x_k-t_k}^{x_k+t_k} g(y)\,\mathrm{d}y-\frac{1}{2}\int_{x-t}^{x+t} g(y)\,\mathrm{d}y\right|\\&\qquad\qquad\le\frac{1}{2}\left|\int_{x_k-t_k}^{x-t} g(y)\,\mathrm{d}y\right|+\frac{1}{2}\left|\int_{x_k+t_k}^{x+t} g(y)\,\mathrm{d}y\right|\xrightarrow{k\to\infty}0\,,\end{aligned}$$

from which we deduce that u is continuous.

Still u is not C^1, but can we say something about the singularities of u? We look at the points of singularity of the initial data f and g and we look at their evolution along the characteristics. Indeed, as observed above, the singularities of the solution propagate along the characteristics.

In our case, singularities are at the points $-1,1$. This means that singularities can only live on the curves

$$\{x+t=1\}\cup\{x-t=1\}\cup\{x+t=-1\}\cup\{x-t=-1\}\,,$$

i.e., $\{x\pm t=1,-1\}$. In particular, we can at least say that u is a generalized solution.

Example 4.2.4. Consider the following Cauchy problem (4.2.1) with $c=2$ and initial data

$$f(x)=g(x)=\begin{cases}1\,, & \text{if } |x|\le 1\\ 0\,, & \text{if } |x|>1\,.\end{cases}$$

We can compute the value of the solution u at $(0,1/2)$ using d'Alembert's formula and we get

$$u(0,1/4)=\frac{f(1/2)+f(-1/2)}{2}+\frac{1}{4}\int_{-1/2}^{1/2} g(s)\,\mathrm{d}s=\frac{5}{4}\,.$$

Moreover, again thanks to d'Alembert's formula, we are able to understand the large time behaviour of the solution u. Indeed, fixed $\bar{x}\in\mathbb{R}$ and let t go to infinity, we have

$$\lim_{t\to\infty}u(\bar{x},t)=\lim_{t\to\infty}\left[\frac{f(\bar{x}+2t)+f(\bar{x}-2t)}{2}+\frac{1}{4}\int_{\bar{x}-2t}^{\bar{x}+2t} g(y)\,\mathrm{d}y\right]\,.$$

For $\bar{x}$ fixed and t sufficiently large, we have $\bar{x}+2t>1$, $\bar{x}-2t<-1$, which means that $f(\bar{x}+2t)=f(\bar{x}-2t)=0$ and $\int_{\bar{x}-2t}^{\bar{x}+2t} g(y)\,\mathrm{d}y=\int_{-1}^{1}\mathrm{d}y=2$. Hence we get

$$\lim_{t\to\infty}u(x,t)=\frac{1}{2}\qquad\text{for all } x\in\mathbb{R}\,.$$

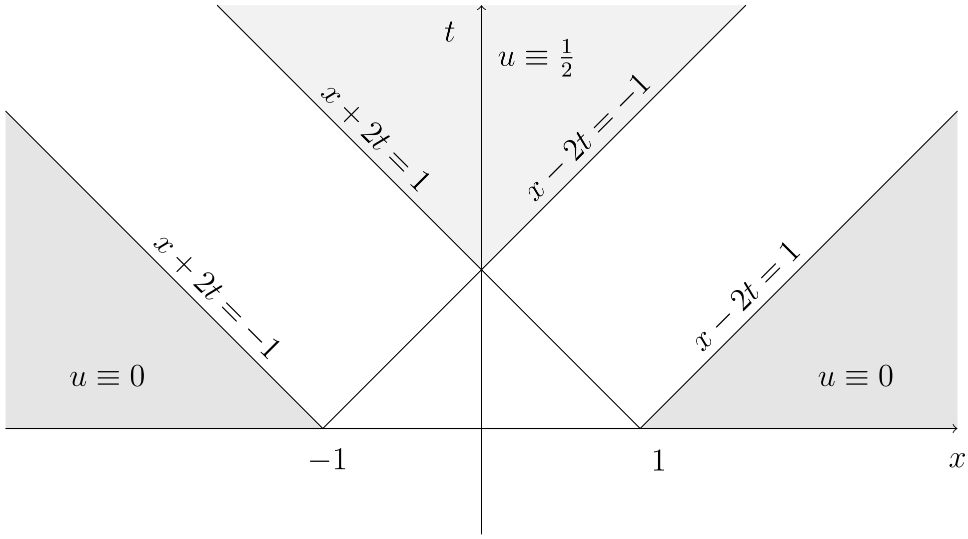

Figure 4.2: Long time behaviour of u.

4.3. Domain of dependence and region of influence

Let us consider the one dimensional homogeneous wave equation as in (4.2.1). How f and g influence the value of u at a given point (x_0, t_0)? How fast the information propagates?

The answer to the second equation is suggested by the factorization of the solution in the sum of two traveling waves as in (4.1.2): the information propagates with finite speed c.

To answer to the first question we recall that, by d'Alembert's formula (4.2.2), the value of u at the point (x_0, t_0) is determined by the values of f at the boundaries of the interval $[x_0 - ct_0, x_0 + ct_0]$ and by the value of g on all the interval. Hence we say that the interval $[x_0 - ct_0, x_0 + ct_0]$ is the *domain of dependence* of u at the point (x_0, t_0). If we change the initial data at points outside the interval, the value of the solution at the point (x_0, t_0) does not change.

Now fix (x_0, t_0) and consider in the (x, t)-plane the characteristic lines (remember, information propagates along characteristics) passing through the point (x_0, t_0), i.e.,

$$x - ct = x_0 - ct_0 \,, \qquad x + ct = x_0 + ct_0 \,.$$

These two lines intersect the x-axis at the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$ respectively. The triangle $\Delta_{(x_0,t_0)}$ formed by these lines and the interval $[x_0 - ct_0, x_0 + ct_0]$ is said *characteristic triangle*, see Figure 4.3.

Remark 4.3.1. If the initial conditions are smooth on $[x_0 - ct_0, x_0 + ct_0]$, the solution itself is smooth in the characteristic triangle $\Delta_{(x_0,t_0)}$.

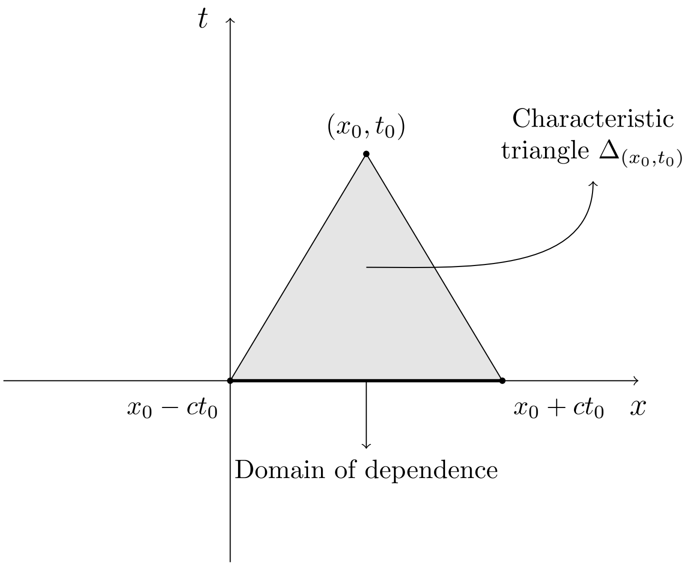

Figure 4.3: The characteristic triangle.

Now we can ask ourselves the dual question: which are the points in the half plane $t > 0$ influenced by the initial data on a fixed interval $[a, b]$? The set of points influenced by the values of f and g in $[a, b]$ is the *region of influence of the interval* $[a, b]$. From d'Alembert's formula and the previous discussion, we discover that the points in $[a, b]$ influence the value of u at a given point (x_0, t_0) if and only if $[x_0 - ct_0, x_0 + ct_0] \cap [a, b] \neq \emptyset$. Hence, the initial conditions along $[a, b]$ influence those points (x, t) that satisfy

$$x - ct \leq b \qquad \text{and} \qquad x + ct \geq a\,,$$

see Figure 4.4.

Remark 4.3.2. If $f = 0$ and $g = 0$ outside $[a, b]$, then the solution u is identically zero to the left of $x + ct = a$ and to the right of $x - ct = b$.

4.4. The Cauchy problem for the nonhomogeneous wave equation

The general nonhomogeneous one dimensional wave equation has the following form

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t), & (x,t) \in \mathbb{R} \times (0, \infty) \\ u(x,0) = f(x)\,, & x \in \mathbb{R} \\ u_t(x,0) = g(x)\,, & x \in \mathbb{R}\,. \end{cases} \tag{4.4.1}$$

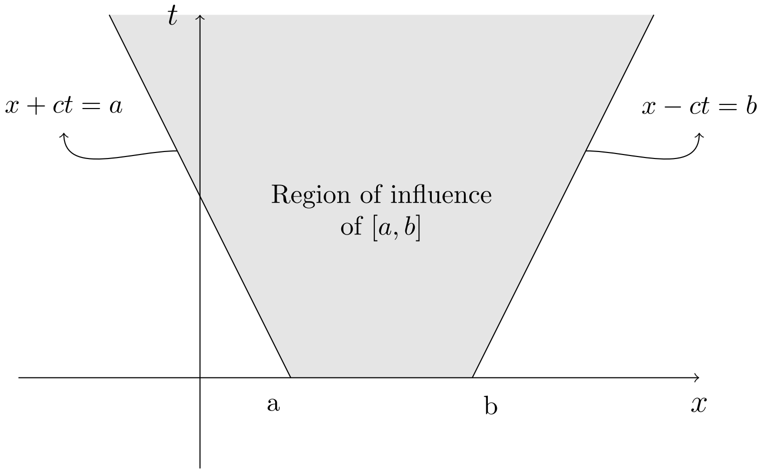

Figure 4.4: The region of influence.

This Cauchy problem models, for example, the vibration of an ideal string subject to an external force $F(x,t)$. As in the homogeneous case, f and g are given functions that represent the shape and the vertical velocity of the string at time zero.

As for the homogeneous case, we wish to have an analogous derivation of d'Alembert's formula. To do this, one integrates over the characteristic triangle $\Delta_{(x_0,t_0)}$ of a generic point (x_0, t_0) and obtains

$$\iint_{\Delta_{(x_0,t_0)}} F(x,t)\,\mathrm{d}x\,\mathrm{d}t = \iint_{\Delta_{(x_0,t_0)}} (u_{tt} - c^2 u_{xx})\,\mathrm{d}x\,\mathrm{d}t\,.$$

Then, after a series of computations, one gets the following theorem.

Theorem 4.4.1. *The solution of the Cauchy problem* (4.4.1) *is given by*

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y + \\ + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau\,,$$

which is d'Alembert's formula for the nonhomogeneous wave equation.

Remark 4.4.2. The value of u at (x_0, t_0) is given by the value of the data f, g, F on the whole characteristic triangle. Note that for $F = 0$ this formula coincides with d'Alembert's formula obtained above.

Example 4.4.3. Consider the following problem

$$\begin{cases} u_{tt} - 4u_{xx} = \sin(x)\,, & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = x\,, & x \in \mathbb{R} \\ u_t(x,0) = x^3\,, & x \in \mathbb{R}\,. \end{cases}$$

Applying d'Alembert's formula with $c = 2$, we get

$$\begin{aligned} u(x,t) &= \frac{x+2t+x-2t}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} s^3 \,\mathrm{d}s + \frac{1}{4}\int_0^t \int_{x-2(t-\tau)}^{x+2(t-\tau)} \sin\xi \,\mathrm{d}\xi\,\mathrm{d}\tau \\ &= x + \frac{1}{16}\left[s^4\right]\Big|_{s=x-2t}^{s=x+2t} - \frac{1}{4}\int_0^t [\cos\xi]\Big|_{\xi=x-2(t-\tau)}^{\xi=x+2(t-\tau)} \mathrm{d}\tau \\ &= x + \frac{1}{16}\left[s^4\right]\Big|_{s=x-2t}^{s=x+2t} - \frac{1}{4}\int_0^t \cos(x+2(t-\tau)) - \cos(x-2(t-\tau))\,\mathrm{d}\tau. \end{aligned}$$

Now, recalling that $\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin(\alpha)\sin(\beta)$ for every angles α and β, we get that

$$\begin{aligned} u(x,t) &= x + \frac{1}{16}\big((x+2t)^2 - (x-2t)^2\big)\big((x+2t)^2 + (x-2t)^2\big) \\ &\quad + \frac{\sin x}{2}\int_0^t \sin(2(t-\tau))\,\mathrm{d}\tau \\ &= x + xt(x^2+4t^2) + \frac{\sin x}{4}[\cos(2(t-\tau))]\Big|_{\tau=0}^{\tau=t} \\ &= x + x^3 t + 4xt^3 + \frac{1}{4}\sin x\big(1-\cos(2t)\big). \end{aligned}$$

Remark 4.4.4. Note that u is an odd function of x. Is this a coincidence?

Example 4.4.5. Consider the nonhomogeneous wave equation given by

$$\begin{cases} u_{tt} - u_{xx} = 2\cos t - t\sin t\,, & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = e^x\,, & x \in \mathbb{R} \\ u_t(x,0) = 0\,, & x \in \mathbb{R}\,. \end{cases}$$

We could solve it using d'Alembert's formula. However, if we can find a particular solution v of the given nonhomogeneous equation, it is possible to reduce the nonhomogeneous problem to a homogeneous one. This eliminates the need to perform the double integral in d'Alembert's formula. This technique is very useful when F is simple, for example when F depends only on x or only on t.

Suppose that we find a particular solution v, then we consider $w := u - v$. Since the wave equation is linear, by superposition principle w solves the following homogeneous problem

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0\,, & (x,t) \in \mathbb{R} \times (0,\infty) \\ w(x,0) = f(x) - v(x,0)\,, & x \in \mathbb{R} \\ w_t(x,0) = g(x) - v_t(x,0)\,, & x \in \mathbb{R}\,. \end{cases}$$

Hence, w can be found using d'Alembert's formula for the homogeneous problem and the final solution is given by $u = v + w$.

Since in our case $F = F(t) = 2\cos t - t\sin t$, we look for a function $v = v(t)$ depending only on t that solves $v_{tt} = 2\cos t - t\sin t$ (note that $v_{xx} = 0$ because v does not depend on x). Let us choose as a particular solution $v(t) = t\sin t$. Of course this solution is not unique because we did not impose any initial condition for v and v_t at time 0. Now define $w(x,t) = u(x,t) - v(t)$ with associated PDE

$$\begin{cases} w_{tt} - w_{xx} = 0\,, & (x,t) \in \mathbb{R} \times (0,\infty) \\ w(x,0) = u(x,0) - v(0) = e^x\,, & x \in \mathbb{R} \\ w_t(x,0) = u_t(x,0) - v_t(0) = 0\,, & x \in \mathbb{R}\,, \end{cases}$$

which is a homogeneous wave equation. Applying d'Alembert's formula we get

$$\begin{aligned} w(x,t) &= \frac{(x+t)e^{x+t} + (x-t)e^{x-t}}{2} = \frac{e^x}{2}((x+t)e^t + (x-t)e^{-t}) \\ &= xe^x \frac{e^t + e^{-t}}{2} + te^x \frac{e^t - e^{-t}}{2} \\ &= xe^x \cosh t + te^x \sinh t\,. \end{aligned}$$

Recalling that $u(x,t) = w(x,t) + v(t)$, this gives

$$u(x,t) = xe^x \cosh t + te^x \sinh t + t\sin t\,.$$

We now prove a uniqueness theorem for the nonhomogeneous one dimensional wave equation (4.4.1).

Theorem 4.4.6. *The problem* (4.4.1) *has a unique solution.*

Proof. The existence of a solution is given by d'Alembert's formula, i.e., Theorem 4.4.1. For the uniqueness, suppose that u_1 and u_2 are solutions of (4.4.1). Then we define the difference $w = u_1 - u_2$, which solves the equation

$$\begin{cases} w_{tt} - c^2 w_{xx} = (u_1)_{xx} - c^2 (u_1)_{xx} - [(u_2)_{xx} - c^2 (u_2)_{xx}] = 0\,, & (x,t) \in \mathbb{R} \times (0,\infty) \\ w(x,0) = u_1(x,0) - u_2(x,0) = f(x) - f(x) = 0\,, & x \in \mathbb{R} \\ w_t(x,0) = (u_1)_t(x,0) - (u_2)_t(x,0) = g(x) - g(x) = 0\,, & x \in \mathbb{R}\,. \end{cases}$$

Hence, by d'Alembert formula (4.2.2), this implies $w(x,t) = 0$ and thus $u_1 = u_2$, as desired. $\square$

4.5. Symmetry of the wave equation

Let us now introduce another property of the wave equation, which shows that the symmetry in Example 4.4.3 was not a fortuity (see Remark 4.4.4).

Theorem 4.5.1. *Given a general nonhomogeneous wave equation, if the initial data f, g and the inhomogeneity F are even (resp. odd, periodic) with respect to x, then the solution is even (resp. odd, periodic) as well.*

Proof. Let us first consider the case in which f, g and F are even with respect to x, i.e., $f(x) = f(-x)$, $g(x) = g(-x)$, $F(x,t) = F(-x,t)$. Let us define $v(x,t) := u(-x,t)$, then we want to prove that $v = u$, namely that u is even with respect to x. We note that

- $v_t(x,t) = u_t(x,t)$, $v_{tt}(x,t) = u_{tt}(-x,t)$;
- $v_x(x,t) = -u_x(-x,t)$, $v_{xx}(x,t) = u_{xx}(-x,t)$.

Therefore it holds

$$\begin{cases} v_{tt} - c^2 v_{xx} = u_{tt}(-x,t) - c^2 u_{xx}(-x,t) = F(-x,t) = F(x,t)\,, & (x,t) \in \mathbb{R} \times (0,\infty) \\ v(x,0) = u(-x,0) = f(-x) = f(x)\,, & x \in \mathbb{R} \\ v_t(x,0) = u_t(-x,0) = g(-x) = g(x)\,, & x \in \mathbb{R}\,, \end{cases}$$

where we used that f, g, F are even. Thus v satisfies the same wave equation with the same boundary conditions as u and therefore $v = u$, by uniqueness Theorem 4.4.6.

The odd case (i.e., $f(x) = -f(-x)$, $g(x) = -g(-x)$ and $F(x,t) = -F(-x,t)$) and the periodic case (i.e., $f(x) = f(x+L)$, $g(x) = g(x+L)$ and $F(x,t) = F(x+L,t)$ for some $L > 0$) can be solved analogously defining $v(x,t) = -u(-x,t)$ and $v(x,t) = u(x+L,t)$, respectively. □

Let us now see how we can apply the previous theorem to solve a particular wave equation with an extra boundary condition. Consider the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0\,, & (x,t) \in (0,\infty) \times (0,\infty) \\ u(x,0) = f(x)\,, & x > 0 \\ u_t(x,0) = g(x)\,, & x > 0 \\ u(0,t) = 0\,, & t \geq 0\,. \end{cases}$$

In order to fulfill the boundary condition $u(0,t) = 0$, we extend f and g in and odd way as

$$\tilde{f}(x) := \begin{cases} f(x)\,, & \text{if } x \geq 0 \\ -f(-x)\,, & \text{if } x < 0\,, \end{cases} \qquad \tilde{g}(x) := \begin{cases} g(x)\,, & \text{if } x \geq 0 \\ -g(-x)\,, & \text{if } x < 0\,. \end{cases}$$

Then we solve the equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0\,, & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = \tilde{f}(x)\,, & x \in \mathbb{R} \\ u_t(x,0) = \tilde{g}(x)\,, & x \in \mathbb{R}\,. \end{cases}$$

The solution u is odd in x because $\tilde{f}$ and $\tilde{g}$ are odd, therefore u satisfies $u(0,t) = 0$. Indeed $u(x,t) = -u(-x,t)$ implies that $u(0,t) = -u(0,t)$ and thus $u(0,t) = 0$.

CHAPTER 5

SEPARATION OF VARIABLES

We now introduce the method of separation of variables to solve linear partial differential equations with boundary and/or initial conditions. Let us directly present the method in the case of the heat equation.

5.1. Heat equation with Dirichlet boundary conditions

The heat equation is a linear second order PDE that describes how the distribution of heat evolves over time in a medium. Heat flows from places where it is higher towards places where it is lower (by the second law of thermodynamics). This equation was derived and solved by Joseph Fourier in 1822.

The heat equation in $\mathbb{R}^3$ is $u_t = k\Delta u$, where $k \in \mathbb{R}$ is the diffusivity of the medium and the function $u = u(t, x, y, z)$ represents the temperature at point (x, y, z) at time t.

The equation says that the rate u_t at which the material at a point (x, y, z) heats up (or cool down) is proportional to how much hotter (or cooler) the surrounding material is. The heat equation arises in the modeling of a number of phenomena, for example

- in financial mathematics, in the modeling of options;
- in probability theory it is connected with the study of the Brownian motion;
- in physics for modeling particle diffusion.

Consider the Cauchy problem associated to the one dimensional heat equation

$$\begin{cases} u_t - ku_{xx} = 0\,, & (x,t) \in (0,L) \times (0,\infty) \\ u(0,t) = u(L,t) = 0\,, & t > 0 \\ u(x,0) = f(x)\,, & x \in (0,L)\,, \end{cases}$$

where $k \in \mathbb{R}^+$ is the constant of diffusivity. This heat equation describes the diffusion of heat in a one dimensional structure (for example a metal bar of length

L) over time, knowing that the initial temperature is equal to f. The boundary conditions are telling us that the boundary of the metal bar is kept at zero, see Figure 5.1. This Cauchy problem is also called an initial boundary problem (and it is homogeneous).

Remark 5.1.1. In order to have compatibility between boundary and initial conditions we assume that $f(0) = f(L) = 0$.

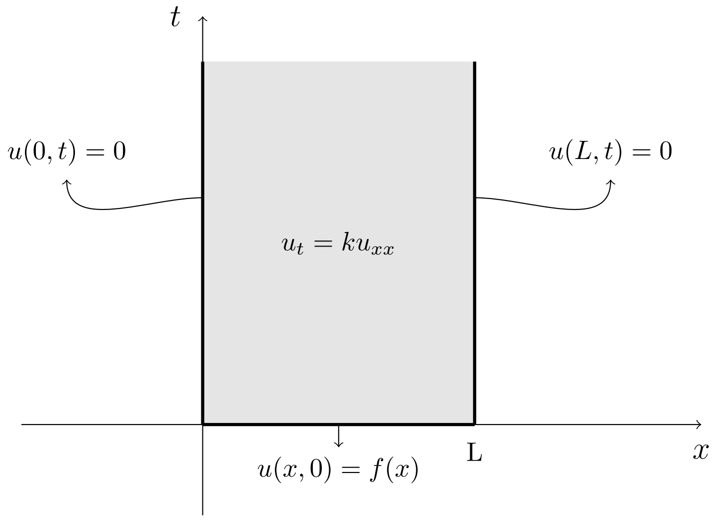

Figure 5.1: The boundary conditions for the heat equation modelled on a metal bar of length L.

Let us now solve this problem using the method of separation of variables. The first step consists in seeking for a solution that has the form of a product solution, or separate solution, i.e.,

$$u(x,t) = X(x)T(t)\,,$$

where $X\colon [0,L] \to \mathbb{R}$, $T\colon [0,\infty) \to \mathbb{R}$. Note that at this step we are not asking that u satisfies the initial condition, but only the boundary conditions.

Plugging this into the heat equation, we get

$$T'(t)X(x) - kX''(x)T(t) = 0 \iff \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}\,.$$

Note that the term on the left-hand side only depends on t, while the term on the right hand side only depends on x. Therefore, the only possibility is that these two functions are equal to a constant $-\lambda$, namely

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda\,.$$

We are now left with two ODEs

$$\begin{cases} X''(x) = -\lambda X(x)\,, & x \in (0, L) \\ T'(t) = -k\lambda T(t)\,, & t > 0\,. \end{cases}$$

These ODEs are only coupled by the separation constant $-\lambda$. Moreover note that u satisfies the boundary conditions $u(0,t) = u(L,t) = 0$ if and only if $u(0,t) = X(0)T(t) = 0$ and $u(L,t) = X(L)T(t) = 0$ for all $t > 0$. These conditions are fulfilled either if $T(t) = 0$ (which gives a trivial solution) or if $X(0) = X(L) = 0$, which represents the interesting case.

Let us now first consider the ODE in X

$$\begin{cases} X''(x) = -\lambda X(x)\,, & x \in (0, L) \\ X(0) = X(L) = 0\,. \end{cases} \tag{5.1.1}$$

We have to distinguish different cases:

- If $\lambda < 0$, the solution has the form

  $$X(x) = \alpha \cosh(\sqrt{-\lambda}x) + \beta \sinh(\sqrt{-\lambda}x)\,.$$

  From the boundary condition $X(0) = 0$, since $\sinh(0) = 0$ and $\cosh(0) = 1$, we get $0 = X(0) = \alpha$. Thus $X(x) = \beta \sinh(\sqrt{-\lambda}x)$. From the boundary condition $X(L) = 0$, we obtain $0 = X(L) = \beta \sinh(\sqrt{-\lambda}L)$. Observe that sinh only vanishes at 0, thus $\sinh(\sqrt{-\lambda}L) \neq 0$ and we get $\beta = 0$. As a result, the only solution compatible with the boundary conditions in this case is the trivial one.

- If $\lambda = 0$, the solution is $X(x) = \alpha + \beta x$. Similarly as before, thanks to the boundary conditions $X(0) = X(L) = 0$ we get $\alpha = \beta = 0$. Again the only possible solution is the trivial one.

- If $\lambda > 0$, the solution for X is

  $$X(x) = \alpha \cos(\sqrt{\lambda}x) + \beta \sin(\sqrt{\lambda}x)\,.$$

  From the boundary condition $X(0) = 0$ we get $\alpha = 0$ and $X(x) = \beta \sin(\sqrt{\lambda}x)$. Hence, in order for X to satisfy $X(L) = 0$, we must have $\sqrt{\lambda}L = n\pi$ for some $n \in \mathbb{N}$, because the sine vanishes only at integer multiples of π. Therefore we get

  $$\lambda = \left(\frac{n\pi}{L}\right)^2\,.$$

  Thus the solutions compatible with the boundary conditions are $X(x) = \beta \sin(n\pi x/L)$ for every $n \in \mathbb{N}$.

Hence we obtain that the set of solutions of (5.1.1) is an infinite sequence of functions

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right), \qquad n \in \mathbb{N}.$$

Now let us consider the problem for T, which is

$$T'(t) = -k\lambda T(t), \qquad t \in (0, \infty).$$

The general solution for this equation is $T(t) = Be^{-k\lambda t}$. Since (5.1.1) has nontrivial solution only for $\lambda_n := (n\pi/L)^2$ with $n \in \mathbb{N}$, we are interested only in the solutions T of the form

$$T_n(t) = B_n e^{-k\lambda_n t},$$

for some $n \in \mathbb{N}$.

We thus obtain a sequence of separated solutions to the heat equation given by

$$u_n(x,t) = X_n(x)T_n(t) = B_n \sin(\sqrt{\lambda_n}x)e^{-k\lambda_n t}.$$

Note that the heat equation is linear, then by superposition principle (see Theorem 1.4.8) any finite linear combination

$$u(x,t) = \sum_{n=1}^{N} B_n \sin(\sqrt{\lambda_n}x)e^{-k\lambda_n t}$$

is still a solution to the heat equation that satisfies the boundary conditions.

At this point we can consider the initial condition. If $f(x)$ admits the following Fourier expansion

$$f(x) = \sum_{n=1}^{\infty} C_n \sin(\sqrt{\lambda_n}x),$$

then a natural candidate for a solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(\sqrt{\lambda_n}x)e^{-k\lambda_n t}.$$

But how to obtain the coefficients C_n from f? Fix $m \in \mathbb{N}$ and multiply the expansion for $f(x)$ by $\sin(m\pi x/L) = \sin(\sqrt{\lambda_m}x)$, obtaining

$$f(x)\sin\left(\sqrt{\lambda_m}x\right) = \sum_{n=1}^{\infty} C_n \sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_m}x).$$

Integrating over $[0, L]$ we thus get

$$\int_0^L f(x)\sin\left(\sqrt{\lambda_m}x\right)\,\mathrm{d}x = \sum_{n=1}^{\infty} C_n \int_0^L \sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_m}x)\,\mathrm{d}x = \frac{L}{2}C_m,$$

where we used that

$$\int_0^L \sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_m}x)\,\mathrm{d}x = \begin{cases} L/2\,, & \text{if } m = n \\ 0\,, & \text{if } m \neq n\,. \end{cases}$$

As a result, we have that

$$C_m = \frac{2}{L}\int_0^L f(x)\sin\left(\sqrt{\lambda_m}x\right)\,\mathrm{d}x$$

for all $m \in \mathbb{N}$. In particular the coefficients are uniquely determined by the initial condition f.

Example 5.1.2. Consider the Cauchy problem

$$\begin{cases} u_t - u_{xx} = 0\,, & (x,t) \in [0,\pi] \times [0,\infty) \\ u(0,t) = u(\pi,t) = 0\,, & t \geq 0 \\ u(x,0) = f(x)\,, & x \in [0,\pi]\,, \end{cases}$$

where $f(x) = \pi x - x^2$. This is a one dimensional heat equation with Dirichlet boundary conditions in $[0,\pi]$. Therefore we know that the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)e^{-n^2 t}\,.$$

Imposing $u(x,0) = f(x)$, we note (as before) the the coefficients B_n are the Fourier coefficients of f, which can we obtained as follows: first of all, by integration by parts we observe that

$$\int_a^b x\sin(nx)\,\mathrm{d}x = \left[-x\frac{\cos(nx)}{n}\right]_a^b + \int_a^b \frac{\cos(nx)}{n}\,\mathrm{d}x = \left[-x\frac{\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_a^b,$$

and

$$\begin{aligned} \int_a^b x^2\sin(nx)\,\mathrm{d}x &= \left[-x^2\frac{\cos(nx)}{n}\right]_a^b + \int_a^b 2x\frac{\cos(nx)}{n}\,\mathrm{d}x \\ &= \left[-x^2\frac{\cos(nx)}{n} + 2x\frac{\sin(nx)}{n^2}\right]_a^b - 2\int_a^b \frac{\sin(nx)}{n^2}\,\mathrm{d}x \\ &= \left[-x^2\frac{\cos(nx)}{n} + 2x\frac{\sin(nx)}{n^2} + 2\frac{\cos(nx)}{n^3}\right]_a^b, \end{aligned}$$

so that the coefficients B_n are

$$\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\pi\Big(-\pi\frac{\cos(n\pi)}{n}\Big) - \frac{2}{\pi}\Big(-\pi^2\frac{\cos(n\pi)}{n} + 2\frac{\cos(n\pi)}{n^3} - \frac{2}{n^3}\Big) \\
&= \frac{4}{\pi n^3} - \frac{4}{\pi n^3}\cos(n\pi) = \begin{cases} 0, & \text{if } n = 2j \text{ for some } j \in \mathbb{N}, \\ \frac{8}{\pi n^3}, & \text{if } n = 2j-1 \text{ for some } j \in \mathbb{N}. \end{cases}
\end{aligned}$$

The solution is thus given by

$$u(x,t) = \sum_{j=1}^{\infty} \frac{8}{\pi(2j-1)^3}\sin((2j-1)x)e^{-(2j-1)^2 t}.$$

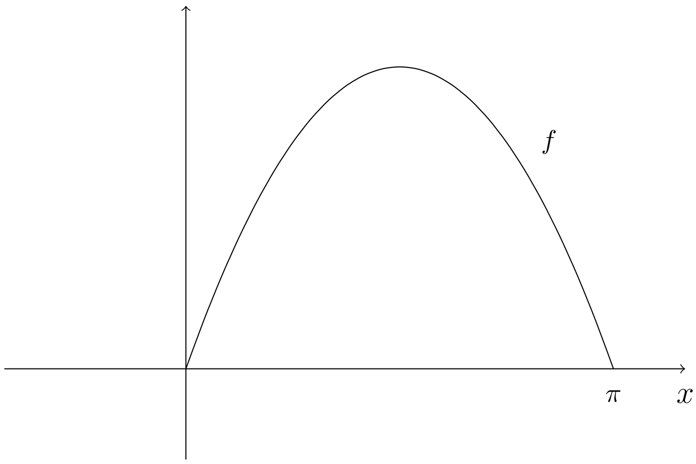

Figure 5.2: The boundary conditions for the wave equation of Example 5.1.2.

5.2. Wave equation with Neumann boundary conditions

So far we discussed the method of separation of variables for the heat equation with Dirichlet boundary conditions. However, later we will encounter three types of boundary conditions:

- Dirichlet: $u(0,t) = u(L,t) = 0$.
- Neumann: $u_x(0,t) = u_x(L,t) = 0$.
- Mixed type, or Robin: $\alpha_0 u(0,t) + \beta_0 u_x(0,t) = \gamma_0$ and $\alpha_L u(L,t) + \beta_L u_x(L,t) = \gamma_L$.

As an example with Neumann boundary conditions, we now present the method of separation of variables applied to the one dimensional wave equation. Consider the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0\,, & (x,t) \in [0,L] \times [0,\infty) \\ u_x(0,t) = u_x(L,t) = 0\,, & t > 0 \\ u(x,0) = f(x)\,, & x \in \mathbb{R} \\ u_t(x,0) = g(x)\,, & x \in \mathbb{R}\,. \end{cases}$$

As before, we look for a solution u not identically zero of the form

$$u(x,t) = X(x)T(t)\,.$$

At this stage we do not take into account the initial conditions. Differentiating in x and t we get $u_{tt} = X(x)T''(t)$ and $u_{xx} = X''(x)T(t)$. Hence, plugging into the equation, we obtain

$$X(x)T''(T) = c^2 X''(x)T(t) \iff \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda\,,$$

for some $\lambda \in \mathbb{R}$. Therefore we have the following ODEs

$$\begin{cases} X''(x) = -\lambda X(x)\,, \quad X'(0) = X'(L) = 0 \\ T''(t) = -c^2 \lambda T(t)\,. \end{cases}$$

The general solution for the ODE in X is

$$X(x) = \begin{cases} \alpha \cosh(\sqrt{-\lambda}x) + \beta \sinh(\sqrt{-\lambda}x)\,, & \text{if } \lambda < 0 \\ \alpha + \beta x\,, & \text{if } \lambda = 0 \\ \alpha \cos(\sqrt{\lambda}x) + \beta \sin(\sqrt{\lambda}x)\,, & \text{if } \lambda > 0\,. \end{cases}$$

Since we need to impose the Neumann boundary conditions $X'(0) = X'(L) = 0$, let us compute X', which is

$$X'(x) = \begin{cases} \sqrt{-\lambda}\, \left[\alpha \sinh(\sqrt{-\lambda}x) + \beta \cosh(\sqrt{-\lambda}x)\right], & \text{if } \lambda < 0 \\ \beta, & \text{if } \lambda = 0 \\ \sqrt{\lambda}\, \left[-\alpha \sin(\sqrt{\lambda}x) + \beta \cos(\sqrt{\lambda}x)\right], & \text{if } \lambda > 0. \end{cases}$$

Let us now consider the three cases separately:

- If $\lambda < 0$, then $X'(0) = 0$ implies $\beta = 0$. Thus, from $X'(L) = 0$, we get $\alpha \sinh(\sqrt{-\lambda}L) = 0$ and then $\alpha = 0$. Therefore $X(x) = 0$, which means that in this case we do not have nontrivial solutions.

- If $\lambda = 0$, the only nontrivial solution is given by $X(x) = X_0(x) = \alpha$.
- If $\lambda > 0$, imposing the boundary conditions we find $\beta = 0$ and $\sin(\sqrt{\lambda}L) = 0$. Thus $\sqrt{\lambda}L = n\pi$ for some $n \in \mathbb{N}$. As a result, $\lambda_n = (n\pi/L)^2$ are eigenvalues for every $n \in \mathbb{N}$ and the corresponding eigenfunctions are $X_n(x) = \alpha_n \cos(\sqrt{\lambda_n}x)$.

Let us now consider the ODE for T, that is $T''(t) = -c^2\lambda T(t)$. If $\lambda = 0$, we get $T_0(t) = \gamma_0 + \delta_0 t$ for some $\gamma_0, \delta_0 \in \mathbb{R}$. On the other hand, for $\lambda = \lambda_n > 0$ the ODE is $T_n''(t) = -c^2\sqrt{\lambda_n}T_n$, with solution

$$T_n(t) = \gamma_n \cos(ct\sqrt{\lambda_n}) + \delta_n \sin(ct\sqrt{\lambda_n})\,,$$

for some $\gamma_n, \delta_n \in \mathbb{R}$.

In conclusion, the general solution for the one dimensional wave equation with Neumann boundary conditions can be written as

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} X_n(x)T_n(t) \\ &= \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)\right]\,. \end{aligned} \tag{5.2.1}$$

The factor $1/2$ in the first term is just for convenience.

Remark 5.2.1. Note that in this case we have a cosines expansion instead of a sines one as for the heat equation, because of the Neumann boundary conditions.

To find A_n we exploit that the function at time 0 is equal to $f(x)$, namely

$$f(x) = u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)\,.$$

Fix $m \in \mathbb{N}$, if we multiply the equation above by $\cos(m\pi x/L)$ and we integrate over $[0, L]$, we get

$$\begin{aligned} \int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x = &\int_0^L \frac{A_0}{2}\cos\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x + \\ &+ \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x\,. \end{aligned}$$

Since

$$\int_0^L \cos\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x = \begin{cases} L\,, & \text{if } m = 0 \\ 0\,, & \text{if } m \geq 1 \end{cases}$$

and

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x = \begin{cases} L/2\,, & \text{if } n = m \neq 0 \\ 0\,, & \text{if } n \neq m\,, \end{cases}$$

we obtain

$$A_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x\,.$$

The same procedure can be implemented to find the coefficients B_m, since we have that

$$g(x) = u_t(x,0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\cos\left(\frac{n\pi}{L}x\right)\,,$$

and we get

$$B_0 = \frac{2}{L}\int_0^L g(x)\,\mathrm{d}x\,, \qquad B_m = \frac{2}{cm\pi}\int_0^L g(x)\cos\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x \quad \text{for } m \geq 1\,.$$

Therefore the problem is formally solved.

Example 5.2.2. Consider the wave equation

$$\begin{cases} u_{tt} - 9u_{xx} = 0\,, & (x,t) \in [0,1] \times [0,\infty) \\ u_x(0,t) = u_x(1,t) = 0\,, & t \geq 0 \\ u(x,0) = f(x) = 1 + \cos(3\pi x) + 16\cos(20\pi x)\,, & x \in [0,1] \\ u_t(x,0) = g(x) = 0\,, & x \in [0,1]\,. \end{cases}$$

Thanks to the arguments above, we can write the solution u as in (5.2.1). To find the coefficients of this expression, we impose the initial conditions

$$1 + \cos(3\pi x) + 16\cos(20\pi x) = u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\,.$$

Integrating the left hand and right hand side against $\cos(n\pi x)$ we find immediately that $A_0 = 2$, $A_3 = 1$, $A_{20} = 16$ and $A_m = 0$ for $m \neq 0, 3, 20$. On the other hand, using that $g(x) = 0$, we obtain that $B_m = 0$ for all $m \in \mathbb{N}$. Thus the solution to the Cauchy problem is

$$u(x,t) = 1 + \cos(3\pi x)\cos(9\pi t) + 16\cos(20\pi x)\cos(60\pi t)\,.$$

5.3. Inhomogeneous PDEs

Let us consider the following inhomogeneous heat equation

$$\begin{cases} u_t - ku_{xx} = h(x,t)\,, & (x,t) \in [0,L] \times \mathbb{R}^+ \\ u(0,t) = u(L,t) = 0\,, & t \in \mathbb{R}^+ \\ u(x,0) = f(x)\,, & x \in [0,L]\,. \end{cases}$$

Recall that, using the separation of variables for the homogeneous heat equation with Dirichlet boundary condition, the admissible solutions for the ODE in X are

$$X_n = \alpha_n \sin\left(\frac{n\pi}{L}x\right)\,, \qquad n \in \mathbb{N}\,.$$

Now, instead of solving also the ODE for $T(t)$, we write a general solution as

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{L}x\right)\,,$$

where T_n is arbitrary. Computing the term $u_t - ku_{xx}$, we get

$$u_t - ku_{xx} = \sum_{n=1}^{\infty} \left[T_n'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t)\right] \sin\left(\frac{n\pi}{L}x\right)\,.$$

Thus, we are now left with the problem of finding T_n. Assume that, for every $t \in \mathbb{R}^+$, $c_n(t)$ is the n-th Fourier coefficient of the inhomogeneity $h(\cdot,t)$, namely

$$c_n(t) = \frac{2}{L}\int_0^L h(x,t) \sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x\,.$$

Then we can express $h(x,t)$ as follows

$$h(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin\left(\frac{n\pi}{L}x\right)\,.$$

As a result, the equation $u_t - ku_{xx} = h$ is equivalent to

$$\sum_{n=1}^{\infty} \left[T_n'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t)\right] \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} c_n(t) \sin\left(\frac{n\pi}{L}x\right)\,.$$

Hence we need to impose

$$T_n'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t) = c_n(t)$$

and the initial condition

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin\left(\frac{n\pi}{L}\right) ,$$

which leads to the ODE system

$$\begin{cases} T_n'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t) = c_n(t) \\ T_n(0) = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}\right) f(x)\,\mathrm{d}x . \end{cases}$$

Thanks to the local existence and uniqueness theorem for ODEs, we have the existence of a unique solution $T_n(t)$ for every $n \in \mathbb{N}$. Therefore, the formal solution of our inhomogeneous Cauchy problem is

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{L}x\right) .$$

Remark 5.3.1. As for the homogeneous case, if the boundary conditions are of Neumann's type, we obtain an expansion in term of cosines and there is a summand for $n = 0$. Moreover, in the case of the inhomogeneous wave equation, we have a second order ODE for T_n, complemented with two initial conditions (for $T_n(0)$ and $T_n'(0)$) that are linked to the Fourier expansion of $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

Example 5.3.2. Consider the inhomogeneous wave equation with Neumann boundary conditions

$$\begin{cases} u_{tt} - u_{xx} = 4\pi^2 \cos(2\pi x)t , \quad (x,t) \in (0,1) \times \mathbb{R}^+ \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = 1 + \cos(2\pi x) \\ u_t(x,0) = 3\cos(2\pi x) . \end{cases}$$

From the method of separation of variables for the homogeneous wave equation with Neumann boundary conditions we have that the admissible solutions for the ODE in X are

$$X_n(x) = \cos(n\pi x) , \qquad n \geq 0 .$$

Hence we try to look for solutions of the form

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t) \cos(n\pi x) .$$

Imposing that $u(x,t)$ satisfies $u_{tt} - u_{xx} = 4\pi^2\cos(2\pi x)t$, we obtain

$$u_{tt} - u_{xx} = \sum_{n=0}^{\infty}[T_n''(t) + n^2\pi^2 T_n(t)]\cos(n\pi x) = 4\pi^2\cos(2\pi x)t\,.$$

Hence we require that

$$\begin{cases} T_2''(t) + 4\pi^2 T_2(t) = t\,, & \text{for } n = 2 \\ T_n''(t) + n^2\pi^2 T_n(t) = 0\,, & \text{for } n \neq 2\,. \end{cases}$$

Since $u(x,0) = 1 + \cos(2\pi x)$ and $u_t(x,0) = 3\cos(2\pi x)$ contain only summands of the form $\cos(n\pi x)$ for $n = 0$ and $n = 2$, we need to consider separately the cases $n = 0, 2$. In particular we have the following three cases:

$$\text{case } n = 0\text{:} \quad \begin{cases} T_0''(t) = 0 \\ T_0(0) = 1 \\ T_0'(0) = 0 \end{cases} \implies T_0(t) = 1$$

$$\text{case } n \neq 0, 2\text{:} \quad \begin{cases} T_n''(t) + n^2\pi^2 T_n(t) = 0 \\ T_n(0) = 0 \\ T_n'(0) = 0 \end{cases} \implies T_n(t) = 0$$

$$\text{case } n = 2\text{:} \quad \begin{cases} T_2''(t) + 4\pi^2 T_2(t) = 4\pi^2 t \\ T_2(0) = 1 \\ T_2'(0) = 3 \end{cases}$$

$$\implies T_2(t) = c_1\sin(2\pi t) + c_2\cos(2\pi t) + t\,,$$

which gives $T_2(0) = c_1 = 1$ and $T_2'(0) = 2\pi c_2 + 1 = 3$. Finally, we thus obtain

$$u(x,t) = 1 + \left[\sin(2\pi t) + \frac{1}{\pi}\cos(2\pi t) + t\right]\cos(2\pi x)\,.$$

Example 5.3.3. Consider the inhomogeneous wave equation

$$\begin{cases} u_{tt} - u_{xx} = \sin(m\pi x)\sin(\omega t)\,, & (x,t) \in (0,1)\times(0,\infty) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0\,, \end{cases}$$

for some $m \in \mathbb{N}$, $\omega \in \mathbb{R}$. We consider solutions of the form $u(x,t) = X(x)T(t)$, as before. The admissible solutions for the ODE in X are

$$X_n(x) = \sin(n\pi x)\,, \qquad n \geq 1\,,$$

and, as before, we look for solutions of the form $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$. Plugging this into the equation we have

$$u_{tt} - u_{xx} = \sum_{n=1}^{\infty} [T_n''(t) + n^2\pi^2 T_n(t)]\sin(n\pi x) = \sin(m\pi x)\sin(\omega t)\,.$$

The ODEs for T_n are given by

$$\begin{cases} T_n''(t) + n^2\pi^2 T_n(t) = 0\,, & \text{if } n \neq m \\ T_m''(t) + m^2\pi^2 T_m(t) = \sin(\omega t) & \\ T_n(0) = 0\,, & \text{for all } n \\ T_n'(0) = 0\,, & \text{for all } n\,. \end{cases}$$

Thus, for $n \neq m$, we have $T_n(t) = 0$, while for $n = m$ we get

$$T_m(t) = a_m\cos(m\pi t) + b_m\sin(m\pi t) + c_m\sin(\omega t)\,,$$

provided $\omega \neq m\pi$. Using the initial conditions $T_m(0) = 0$ and $T_m'(0) = 0$, we obtain

$$T_m(t) = \frac{1}{\omega^2 - m^2\pi^2}\left(\frac{\omega}{m\pi}\sin(m\pi t) - \sin(\omega t)\right)\,,$$

and the solution $u(x,t)$ is finally given by

$$u(x,t) = \frac{1}{\omega^2 - m^2\pi^2}\left(\frac{\omega}{m\pi}\sin(m\pi t) - \sin(\omega t)\right)\sin(m\pi x)\,.$$

Remark 5.3.4. We are assuming $\omega \neq m\pi$ to avoid degeneracy. To deal with the case $\omega = \pi m$, we can think it as limit case as $\omega \neq \pi m$, $\omega \to m\pi$. Then

$$\lim_{\omega\to m\pi} u(x,t) = \frac{1}{2m\pi}\left(\frac{\sin(m\pi t)}{m\pi} - t\cos(m\pi t)\right)\sin(m\pi x).$$

Note that, if $\omega \neq \pi m$ for all $m \in \mathbb{N}$, the solution is bounded. In other words, a bounded periodic force with time frequency ω different from the frequencies of the homogeneous solutions produces bounded oscillations. On the other hand, for $\omega = m\pi$ for some $m \in \mathbb{N}$, the solution is unbounded. This is called *resonance effect* (see collapse of the Tacoma bridge).

5.4. Uniqueness with the energy method

One of the main applications of the energy method is the proof of uniqueness for solutions of initial boundary value problems. This method is based on the physical principle of conservation of energy, although the quantity we refer as "energy" may differ from the actual physical energy of the system. We illustrate the method in the following example.

Consider the inhomogeneous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t)\,, & (x,t) \in [0,L] \times \mathbb{R}^+ \\ u_x(0,t) = a(t) \\ u_x(L,t) = b(t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x)\,. \end{cases}$$

Let u_1, u_2 be solutions and set $w := u_1 - u_2$. Then w solves

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0\,, & (x,t) \in [0,L] \times \mathbb{R}^+ \\ w_x(0,t) = w_x(L,t) = 0 \\ w(x,0) = 0 \\ w_t(x,0) = 0\,. \end{cases}$$

Let us define the energy function

$$E(t) := \int_0^L (w_t(t,x))^2 + c^2 (w_x(t,x))^2 \,\mathrm{d}x\,.$$

By taking the derivative of $E(t)$ we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E(t) &= \int_0^L (2 w_t w_{tt} + 2c^2 w_x w_{xt}) \,\mathrm{d}x \\ &= 2 \int_0^L (w_t w_{tt} - c^2 w_{xx} w_t) \,\mathrm{d}x + [2c^2 w_x w_t]\big|_0^L = 0\,. \end{aligned}$$

Therefore $E(t)$ is constant, and, since $E(0) = 0$, it follows that $E(t) = 0$ for all t. By looking at the definition of $E(t)$, we realize that $E(t) = 0$ for all t implies that $w_x(x,t) = w_t(x,t) = 0$ for all x, t, thus w is constant too. Using that $w(x,0) = 0$ for all x, we then get $w(x,t) = 0$. Thus $u_1 \equiv u_2$, which proves uniqueness.

CHAPTER 6

ELLIPTIC EQUATIONS

In this chapter we study Laplace's and Poisson's equations, which are the archetype of elliptic equations. We examine the main properties of elliptic equations, the link between solutions of Laplace's equation and harmonic functions.

6.1. Classification of linear second order PDEs

Assuming that $u_{xy} = u_{yx}$ (which is always the case in these notes), any general linear second order PDE in two independent variables has the form

$$\mathcal{L}[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g\,.$$

The term $au_{xx} + 2bu_{xy} + cu_{yy}$ is the *leading term*, or *principal part*, because the behaviour of the PDE is determined by a, b and c.

Remark 6.1.1. The coefficients a, b and c depends on x, y, i.e., $a = a(x, y)$, $b = b(x, y)$ and $c = c(x, y)$.

As we already said, being able to properly classify the PDE we wish to investigate allows us to choose the correct method (if it exists!) to tackle the PDE. Knowing the "type" of the equation allows one to use the relevant methods to solve it, which can be quite different depending on the type of the equation.

You probably encountered conic sections and quadratic forms, which are usually classified into parabolic, elliptic and hyperbolic, based on the discriminant $b^2 - 4ac$. The same can be done for a second order PDE at a given point.

Given a point (x_0, y_0), consider the value

$$\delta(\mathcal{L})(x_0, y_0) := b^2(x_0, y_0) - a(x_0, y_0)c(x_0, y_0)\,.$$

At the point (x_0, y_0) the PDE is said to be

- *hyperbolic* if $\delta(\mathcal{L})(x_0, y_0) > 0$;
- *parabolic* if $\delta(\mathcal{L})(x_0, y_0) = 0$;

- *elliptic* if $\delta(\mathcal{L})(x_0, y_0) < 0$.

Remark 6.1.2. Since there is the convention that the xy term is $2b$, then the discriminant becomes $(2b)^2 - 4ac = 4(b^2 - ac) = 4\delta[\mathcal{L}]$ (and the 4 can be dropped).

Remark 6.1.3. This classification describes a local property. However, we often study PDEs with constant coefficients, where the classification is global.

Example 6.1.4. Consider the following PDEs:

- The wave equation $u_{tt} - u_{xx} = 0$ is hyperbolic (we use variables (x, t) instead of (x, y)).
- The heat equation $u_t - u_{xx} = 0$ is parabolic.
- Laplace's equation $u_{xx} + u_{yy} = 0$ is elliptic (here we use (x, y) intended as spatial variables).

Similarly to what happens with second order algebraic equations, we can use a nondegenerate change of variables to reduce the equation to a simpler form.

Definition 6.1.5. A transformation $(x, y) \mapsto (\xi, \eta) = (\xi(x, y), \eta(x, y))$ is a *change of coordinates* near a point (x_0, y_0) if

$$\det \begin{pmatrix} \partial_x \xi & \partial_y \xi \\ \partial_x \eta & \partial_y \eta \end{pmatrix} \Big|_{(x,y)=(x_0,y_0)} \neq 0\,.$$

Any second order PDE can be transformed in the so-called *canonical form* by using a change of coordinates $u(x, y) \mapsto w(\xi, \eta) = w(\xi(x, y), \eta(x, y))$. The canonical forms are:

- hyperbolic: $w_{\xi\eta} + \tilde{d}w_\xi + \tilde{e}w_\eta + \tilde{f}w = \tilde{g}$;
- parabolic: $w_{\xi\xi} + \tilde{d}w_\xi + \tilde{e}w_\eta + \tilde{f}w = \tilde{g}$;
- elliptic: $w_{\xi\xi} + w_{\eta\eta} + \tilde{d}w_\xi + \tilde{e}w_\eta + \tilde{f}w = \tilde{g}$.

Example 6.1.6. Consider the wave equation $u_{tt} - c^2 u_{xx} = 0$ for $t \geq 0$. Let us apply the transformation

$$\begin{cases} \xi = x + ct \\ \eta = x - ct\,. \end{cases}$$

This gives us $u(x, t) = w(\xi, \eta) = w(x + ct, x - ct)$. Plugging this into the wave equation gives $0 = u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta}$. Dividing by $-4c^2$, we get $w_{\xi\eta} = 0$, which is in hyperbolic canonical form.

6.2. Laplace's and Poisson's equations

Poisson's equation $\Delta u = f$ and its homogeneous counterpart, Laplace's equation $\Delta u = 0$, have a very prominent role in applied sciences. For example, the temperature of a homogeneous and isotropic body at equilibrium is a solution of Laplace's equation. In this case we can say that Laplace's equation describes the stationary case (independent of time) of the diffusion equation. Other examples are:

- The equilibrium position of a perfectly elastic membrane solves $\Delta u = 0$.
- Poisson's equation plays an essential role in the theory of conservative fields (electric field, magnetic field, gravitational field, etc). If u is the electrostatic potential, then Poisson's equation $\Delta u = f$ represents the link between the potential u and the charge density $-f$.

6.3. Basic properties of elliptic problems

We study some basic models involving the Laplacian, including models for heat conduction, elasticity, electromagnetism, and gravitation. We consider $u = u(x, y)$ for $(x, y) \in D$, where D is is an open subset of $\mathbb{R}^2$.

Definition 6.3.1. We say that u is *harmonic* if it solves *Laplace's equation*, i.e., $\Delta u(x, y) = u_{xx} + u_{yy} = 0$. The nonhomogeneous version of Laplace's equation is *Poisson's equation* $\Delta u(x, y) = \rho(x, y)$.

Remark 6.3.2. Laplace's and Poisson's equations are second order linear PDEs. Laplace's equation is also homogeneous.

Remark 6.3.3. The linearity of Laplace's operator implies that a linear combination of harmonic functions is a harmonic function.

Definition 6.3.4. Let $D \subset \mathbb{R}^2$ an open set and let ∂D be the boundary of D. Let ν be the unit outward normal to ∂D. Then we can consider the following *Dirichlet problem for Poisson's equation*

$$\begin{cases} \Delta u(x, y) = \rho(x, y)\,, & (x, y) \in D \\ u(x, y) = g(x, y)\,, & (x, y) \in \partial D\,. \end{cases} \tag{6.3.1}$$

On the other hand, the *Neumann problem for Poisson's equation* reads as follows

$$\begin{cases} \Delta u(x, y) = \rho(x, y)\,, & (x, y) \in D \\ \partial_\nu u(x, y) = g(x, y)\,, & (x, y) \in \partial D\,. \end{cases} \tag{6.3.2}$$

Finally we can consider the *problem of the third kind for Poisson's equation*, that is

$$\begin{cases} \Delta u(x,y) = \rho(x,y)\,, & (x,y) \in D \\ u(x,y) + \alpha\partial_\nu u(x,y) = g(x,y)\,, & (x,y) \in \partial D\,, \end{cases} \tag{6.3.3}$$

where α and g are given functions.

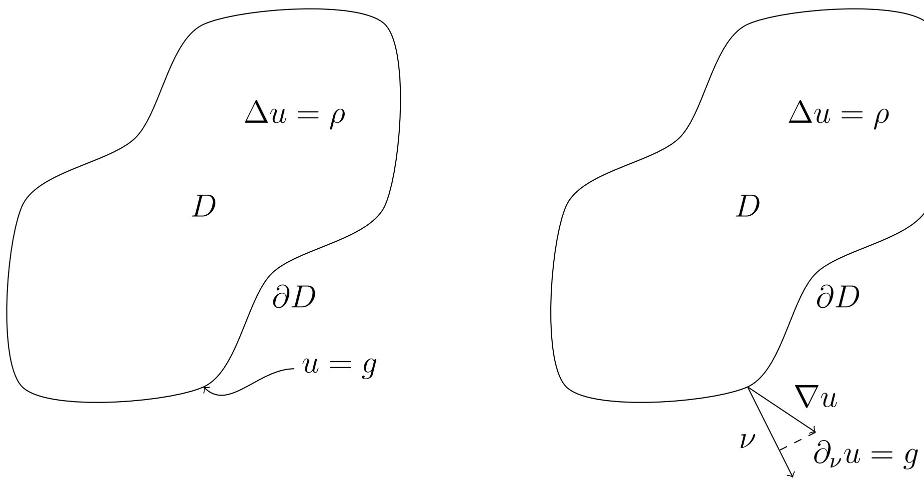

Figure 6.1: Dirichlet and Neumann problems.

We can now ask if a solution to those problem exists. Consider the Neumann problem, which can model the distribution of the temperature $u(x,y)$ in the domain D at an equilibrium configuration. This means that the heat flux through the boundary must be balanced by the temperature production inside the domain. This simple consideration is encoded in the following lemma.

Lemma 6.3.5. *A necessary condition for the existence of a solution to the Neumann problem* (6.3.2) *is*

$$\int_{\partial D} g(x(s), y(s))\,\mathrm{d}s = \int_D \rho(x,y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

where $(x(s), y(s))$ *is a parametrization of* ∂D*.*

Proof. Recall the identity $\Delta u = \operatorname{div}(\nabla u)$. Then Poisson's equation reads as $\operatorname{div}(\nabla u) = \rho$. If u is a solution of the Neumann problem, using Gauss' theorem we have

$$\int_D \rho = \int_D \Delta u = \int_D \operatorname{div}(\nabla u) = \int_{\partial D} \nabla u \cdot \nu = \int_{\partial D} \partial_\nu u = \int_{\partial D} g\,.$$

Therefore $\int_D \rho = \int_{\partial D} g$ as desired. □

Remark 6.3.6. If u is a solution of Laplace's equation $\Delta u = 0$, then we have that $\int_{\partial A} \partial_n u = \int_A \operatorname{div}(\nabla u) = \int_A \Delta u = 0$ for every open subset $A \subset D$, where n is the outward unit normal to ∂A.

An other natural question to ask is if the Cauchy problem for Laplace's equation is well-posed, i.e., if a solution exists, if it is unique and it is stable with respect to the initial conditions. We recall that the Cauchy problem for Laplace's equation is

$$\begin{cases} \Delta u = 0\,, & (x,y) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) \\ u_y(x,0) = g(x)\,, \end{cases}$$

where y here plays the role of time (see also the wave Equation (4.2.1)).

Consider Laplace's equation in the half-plane $x \in \mathbb{R}$, $y > 0$. The following counterexample to well-posedness is due to Hadamard. Consider the following Cauchy problem

$$\begin{cases} \Delta u(x,y) = 0\,, & x \in \mathbb{R}, y > 0 \\ u(x,0) = 0 \\ u_y(x,0) = \sin(nx)/n\,, \end{cases}$$

for some $n \in \mathbb{N}$. We look for solutions of the form

$$u(x,y) = \sin(nx) Y(y)\,.$$

Then, from $\Delta u = 0$, we have

$$0 = u_{xx} + u_{yy} = -n^2 \sin(nx) Y(y) + \sin(nx) Y''(y)\,,$$

which implies that $Y''(y) = n^2 Y(y)$. From the Dirichlet conditions $u(x,0) = 0$ it follows that $Y(0) = 0$, while by the Neumann condition $u_y(x,0) = \sin(nx)/n$ we have

$$\frac{\sin(nx)}{n} = u_y(x,0) = \sin(nx) Y'(0) \implies Y'(0) = \frac{1}{n}\,.$$

Hence, solving the problem for Y we get $Y(y) = \sinh(ny)/n^2$ and thus we obtain the solution of the Cauchy problem

$$u(x,y) = \frac{1}{n^2} \sin(nx) \sinh(ny)\,.$$

Now, setting $u^n(x,y) = \frac{1}{n^2} \sin(nx) \sinh(ny)$, we realize that in the limit $n \to \infty$ both $u^n(x,0)$ and $u^n_y(x,0)$ tend to zero (the initial conditions describe an arbitrary small perturbation of the trivial solution $u = 0$). On the other hand, the solution is not bounded in the half-plane $y > 0$. Indeed, for any $a > 0$, we have

$$\begin{aligned}\sup_{x\in\mathbb{R}} |u^n(x,a)| = \sup_{x\in\mathbb{R}} \frac{1}{n^2}|\sin(nx)|\sinh(na) &= \frac{1}{n^2}\sinh(na) \\ &= \frac{1}{2n^2}(e^{na} - e^{-na}) \xrightarrow{n\to\infty} \infty\,.\end{aligned}$$

Thus, the Cauchy problem for Laplace's equation is not stable and this implies that it is not well-posed with respect to the initial conditions.

In the next example we construct an initial datum for which there is no solution to the Cauchy problem using the Hadamard counterexample.

Example 6.3.7. Consider as before the functions $u^n(x,y) = \sin(nx)\sinh(ny)/n^2$ and define

$$\overline{u}^N(x,y) := \sum_{n=1}^{N} \frac{u^n(x,y)}{n}\,,$$

for which it holds $\overline{u}^N(x,0) = 0$ and

$$\overline{u}^N_y(x,0) = \sum_{n=1}^{N} \frac{u^n_y(x,0)}{n} = \sum_{n=1}^{N} \frac{1}{n^2}\sin(nx)\,.$$

Moreover $\overline{u}^N$ is a solution of Laplace's equation by linearity. However, for N that goes to infinity, we do not have existence to the Cauchy problem

$$\begin{cases} \Delta u(x,y) = 0\,, & (x,y) \in \mathbb{R}\times(0,\infty) \\ u(x,0) = 0 & \\ u_y(x,0) = \sum_{n=1}^{\infty}\sin(nx)/n^2\,, & \end{cases}$$

because the solution would be given by $\overline{u}^\infty := \sum_{n=1}^{\infty} u^n(x,y)/n$, which is not a convergent series. However note that the initial conditions $\overline{u}^\infty(x,0) = 0$ and $\overline{u}^\infty_y(x,0) = \sum_{n=1}^{\infty}\sin(nx)/n^2$ do make perfectly sense.

Remark 6.3.8. These examples demonstrate the difference between elliptic and hyperbolic problems on the upper half-plane.

6.4. Harmonic functions

Let us now compute some harmonic functions. We define *harmonic polynomial* of degree n a harmonic function $P(x,y)$ of the form

$$P(x,y) = \sum_{0\le i+j\le n} a_{ij}x^iy^j\,.$$

For example:

- For $n = 0$, $u(x, y) = 1$.
- For $n = 1$, $u(x, y) = x$ and $u(x, y) = y$, thus in general $u(x, y) = ax + by$ for any $a, b \in \mathbb{R}$.
- For $n = 2$, $u(x, y) = xy$ and $u(x, y) = x^2 - y^2$.
- For $n = 3$, $u(x, y) = x^3 - 3xy^2$ and $u(x, y) = y^3 - 3x^2y$.

CHAPTER 7

MAXIMUM PRINCIPLES

The maximum principle is a fundamental property of solutions to certain PDEs of elliptic or parabolic type. Maximum principles are based on the observation that, if a C^2 function u attains its maximum over an open set D at a point $\mathbf{x}_0 \in D$, then

$$Du(\mathbf{x}_0) = 0\,, \qquad D^2u(\mathbf{x}_0) \leq 0\,,$$

where D^2u is the Hessian matrix. To use this observation, we need to work with solutions that are at least C^2.

7.1. Weak maximum principle

First we identify circumstances under which a function must attain its maximum (or minimum) on the boundary.

Theorem 7.1.1 (Weak maximum principle)**.** *Let D be a bounded domain and let $u(x,y) \in C^2(D) \cap C(\overline{D})$ be a harmonic function in D. Then the maximum of u in $\overline{D}$ is achieved on the boundary ∂D, namely*

$$\max_{\overline{D}} u = \max_{\partial D} u\,.$$

Proof. Consider the function $u_\varepsilon(x,y) = u(x,y) + \varepsilon(x^2+y^2)$, with $\varepsilon > 0$. Assume by contradiction that u_ε attains a local maximum at $(\overline{x},\overline{y}) \in D$. Then, $\Delta u_\varepsilon(\overline{x},\overline{y}) \leq 0$. On the other hand, since u is harmonic, we have that

$$\Delta u_\varepsilon(\overline{x},\overline{y}) = \Delta u(\overline{x},\overline{y}) + 4\varepsilon = 4\varepsilon > 0\,,$$

which is a contradiction. This proves that u_ε takes its maximum on the boundary, $\max_{\overline{D}} u_\varepsilon = \max_{\partial D} u_\varepsilon$. Thus, since $u \leq u_\varepsilon$ and D is bounded, we get

$$\begin{aligned}\max_{\overline{D}} u \leq \max_{\overline{D}} u_\varepsilon = \max_{\partial D} u_\varepsilon = \max_{\partial D}(u + \varepsilon(x^2+y^2))\\ \leq \max_{\partial D} u + \varepsilon \max_{\partial D}(x^2+y^2) = \max_{\partial D} u + \varepsilon c\,.\end{aligned}$$

Letting $\varepsilon \to 0$, it follows that $\max_{\overline{D}} u \leq \max_{\partial D} u$. $\square$

Corollary 7.1.2. *Under the same assumptions of Theorem 7.1.1, we have*

$$\min_{\overline{D}} u = \min_{\partial D} u \,.$$

Proof. Note that $\Delta(-u) = -\Delta u = 0$, hence we can apply Theorem 7.1.1 to $-u$ and obtain

$$\min_{\overline{D}} u = -\max_{\overline{D}}(-u) = -\max_{\partial D}(-u) = \min_{\partial D} u \,.$$

□

Remark 7.1.3. The boundedness in Theorem 7.1.1 and Corollary 7.1.2 of D is necessary. Indeed, if one takes $D = \mathbb{R}^2 \setminus B_1$, then $u(x,y) = \log(x^2+y^2)$ is harmonic in D, $u|_{\partial D} = 0$, but $\sup_D u = \infty \neq 0$.

7.2. Mean value principle

Theorem 7.2.1 (Mean value principle). *Consider a harmonic function u on D and let $B_R(x_0,y_0) \subset D$ be a ball of radius R. Then*

$$\begin{aligned} u(x_0,y_0) &= \frac{1}{2\pi R}\int_{\partial B_R(x_0,y_0)} u(x(s),y(s))\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\,\mathrm{d}\theta \,. \end{aligned} \tag{7.2.1}$$

Proof. Given $r \in (0,R)$, set

$$V(r) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\,\mathrm{d}\theta \,,$$

and compute

$$\begin{aligned} &V'(r) = \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}r} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left[u_x(x_0 + r\cos\theta, y_0 + r\sin\theta)\cos\theta + u_y(x_0 + r\cos\theta, y_0 + r\sin\theta)\sin\theta\right]\mathrm{d}\theta \\ &= \frac{1}{2\pi r}\int_{\partial B_r(x_0,y_0)} \partial_\nu u = \int_{B_r(x_0,y_0)} \Delta u = 0 \,. \end{aligned}$$

As a result, $V(R) = V(0)$, which gives exactly what we want. □

Remark 7.2.2. The inverse implication is also true, namely a smooth function that satisfies the mean value property in some domain D is harmonic in D.

7.3. Strong maximum principle

Theorem 7.3.1 (Strong maximum principle)**.** *Let u be a harmonic function in D, an open connected subset of $\mathbb{R}^2$. If u attains its maximum (or its minimum) at an interior point of D, then u is constant.*

Proof. Let $\mathbf{x}_0 \in D$ be a maximum point for u. Let $\mathbf{x}$ be another point connected to $\mathbf{x}_0$ by a curve γ.

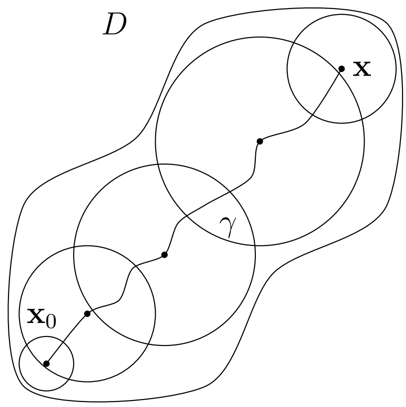

Figure 7.1: Proof of the Strong maximum principle

Choose $R > 0$ smaller than the distance from γ to ∂D and define inductively a sequence of points $\{\mathbf{x}_i\}_{i=0}^N \subset \gamma$ and radii $R_i < R$ such that $\mathbf{x}_{i+1} \in \partial B_{R_i}(\mathbf{x}_i)$ for any $i = 1, \dots, N-1$ and $\mathbf{x}_N = \mathbf{x}$. Note that one can take $R_i = R$ for each $i = 0, \dots, N-2$ and then $R_{N-1} \leq R$ such that $\mathbf{x}_N \in \partial B_{R_{N-1}}(\mathbf{x}_{N-1})$.

Then inside each ball we apply inductively the mean value theorem, Theorem 7.2.1. More precisely, by the mean value theorem applied at $\mathbf{x}_0$ we have

$$\max_D u = u(\mathbf{x}_0) = \frac{1}{2\pi R}\int_{\partial B_R(\mathbf{x}_0)} u \leq \frac{1}{2\pi R}\int_{\partial B_R(\mathbf{x}_0)} \max_D u = \max_D u\,.$$

This implies that $u = \max_D u$ on $\partial B_R(\mathbf{x}_0)$. Therefore, since $\mathbf{x}_1 \in \partial B_R(\mathbf{x}_0)$, also $\mathbf{x}_1$ is a point of maximum for u. Hence we can repeat the argument above (using the mean value theorem) to deduce that $u = \max_D u$ on $\partial B_R(\mathbf{x}_1)$, hence $\mathbf{x}_2 \in \partial B_R(\mathbf{x}_1)$ is a maximum for u, and iterating we get that $\mathbf{x} = \mathbf{x}_N$ is a maximum for u as well. In particular $u(\mathbf{x}_0) = \max_D u = u(\mathbf{x})$. By arbitrariness of $\mathbf{x} \in D$, this proves that $u = \max_D u$ is constant in D. □

Remark 7.3.2. Given a point $(x_0, y_0) \in D$ and a radius $r > 0$, consider the curve $\gamma(\theta) = (x_0 + r\cos\theta, y_0 + r\sin\theta)$.

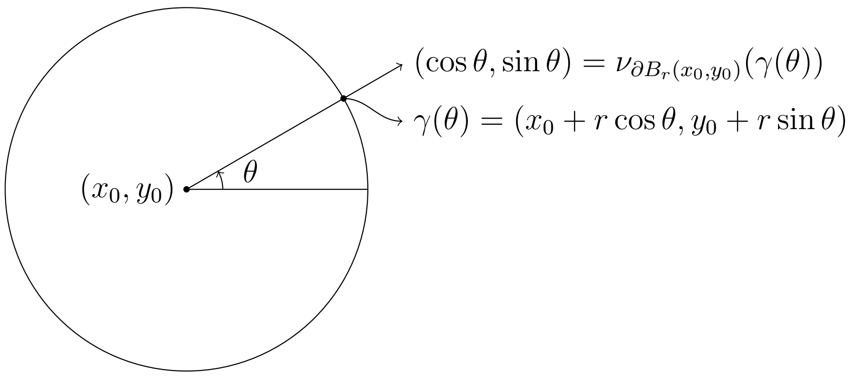

Figure 7.2: Mean value principle on the circle

Then (7.2.1) can be rewritten as

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} \nabla u(\gamma(\theta)) \cdot \nu_{\partial B_r(x_0,y_0)}(\gamma(\theta))\,\mathrm{d}\theta\,.$$

Let us define $F(\gamma(\theta)) = \nabla u(\gamma(\theta)) \cdot \nu_{\partial B_r(x_0,y_0)}(\gamma(\theta))$. Then we have

$$\begin{aligned} u(x_0, y_0) &= \frac{1}{2\pi}\int_0^{2\pi} F(\gamma(\theta))\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|\gamma'(\theta)|} F(\gamma(\theta))|\gamma'(\theta)|\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi r} F(\gamma(\theta))|\gamma'(\theta)|\,\mathrm{d}\theta = \frac{1}{2\pi r}\int_\gamma F\,, \end{aligned}$$

where the second-last equality follows from the fact that $|\gamma'(\theta)| = r$, because $\gamma'(\theta) = (-r\sin\theta, r\cos\theta)$, and the last equality is the definition of integral along a curve.

7.4. Maximum principle for Poisson's equation

Now we examine some important consequences of the maximum principles. Let us assume that the domain D is bounded, then we have the following theorem.

Theorem 7.4.1. *Given a bounded domain $D \subset \mathbb{R}^2$, the Dirichlet problem*

$$\begin{cases} \Delta u = f\,, & \text{in } D \\ u = g'\,, & \text{in } \partial D\,. \end{cases}$$

has at most one solution $u \in C^2(D) \cap C(\overline{D})$.

Proof. Assume by contradiction that there exist two solutions u_1, u_2. Then define $u := u_1 - u_2$, which solves

$$\begin{cases} \Delta u = 0\,, & \text{in } D \\ u = 0\,, & \text{in } \partial D\,. \end{cases}$$

From the weak maximum principle Theorem 7.1.1 we get that the maximum and the minimum of u are zero, which implies $u \equiv 0$ and thus $u_1 \equiv u_2$. □

Remark 7.4.2. In the previous theorem the boundedness condition on D is necessary. The reason for this is the same as for the weak maximum principle. Indeed, let us consider a harmonic function u in $\mathbb{R}^2 \setminus B_1(0)$, with $u = 1$ on $\partial B_1(0)$. Then $u_1 = 1$ is a solution to this problem, but also $u_2 = 1 + \log(x^2 + y^2)$ is a solution. Hence there is no uniqueness.

Theorem 7.4.3. *Let $D \subset \mathbb{R}^2$ be a bounded domain. Let $u_1, u_2 \in C^2(D) \cap C(\overline{D})$ solve $\Delta u_1 = 0$, $\Delta u_2 = 0$ with Dirichlet boundary conditions $u_1 = g_1$ on ∂D and $u_2 = g_2$ on ∂D. Then*

$$\max_{\overline{D}} |u_1 - u_2| = \max_{\partial D} |g_1 - g_2|\,.$$

Proof. Define $v := u_1 - u_2$, then v is harmonic in D and $v = g_1 - g_2$ on ∂D. Therefore the maximum principle Theorem 7.1.1 implies

$$\max_{\overline{D}} v = \max_{\partial D} v = \max_{\partial D} (g_1 - g_2)\,,$$

and by the minimum principle Corollary 7.1.2 we have

$$\min_{\overline{D}} v = \min_{\partial D} v = \min_{\partial D} (g_1 - g_2)\,.$$

Therefore $\max_{\overline{D}} |u_1 - u_2| = \max_{\partial D} |g_1 - g_2|$ as desired. □

7.5. Boundary conditions

We recall some different types of boundary conditions.

- Dirichlet: $u = g$ on ∂D.

It may be referred also as *condition of first type* or as a *fixed boundary condition.* For example the following would be considered Dirichlet conditions:

(a) In thermodynamics when a surface or an object is held at a fixed temperature.

(b) In electromagnetism when a node of a circuit is held at a fixed voltage.

(c) In fluid dynamics, the no-slip condition for viscous fluids states that at a solid boundary the fluid has zero velocity relative to the boundary.

- Neumann: $\partial_\nu u = g$ on ∂D, where ν is the outer normal vector to D.

This is also called *second type boundary condition* and it specifies the values in which the derivative of a solution is applied within the boundary of the domain. An application in thermodynamics is a prescribed heat flux from a surface, which serves as boundary condition. For example, a perfect insulator has no flux, while an electrical component may be dissipating at a known power.

- Robin or third type boundary condition: $u + \alpha\partial_\nu u = g$ on ∂D, where $\alpha \in \mathbb{R}$

and g is given function.
Robin boundary conditions are also called *impedance boundary conditions* from their application in electromagnetic problems.

7.6. Maximum principle for parabolic equations

The maximum principle holds also for parabolic equations. Consider the heat equation for $u = u(t, \mathbf{x})$, $t > 0$, $\mathbf{x} \in D$, namely

$$u_t = k\Delta u\,.$$

Define the domain $Q_T := [0, T] \times D$, where D is the spatial domain and $t \in [0, T]$ is the time. Then we define the *parabolic boundary* as

$$\partial_P Q_T := \{\{0\} \times D\} \cup \{[0, T] \times \partial D\}\,,$$

that is the boundary of Q_T except for the top cover $\{T\} \times D$.

Theorem 7.6.1 (Maximum principle for the heat equation)**.** *Let u solve the homogeneous heat equation $u_t = k\Delta u$ in $Q_T = [0, T] \times D$ for some $k > 0$. Assume that $D \subset \mathbb{R}^2$ is bounded. Then u achieves its maximum (and minimum) on $\partial_P Q_T$.*

Proof. Take $\varepsilon > 0$ and consider the function $u_\varepsilon(t, \mathbf{x}) = u(t, \mathbf{x}) - \varepsilon t$. Then $\partial_t u_\varepsilon = \partial_t u - \varepsilon$, and $\Delta u_\varepsilon = \Delta u$, therefore $\partial_t u_\varepsilon = k\Delta u_\varepsilon - \varepsilon$. Assume by contradiction that u_ε has a maximum at some point $(t_0, \mathbf{x}_0) \in Q_T \setminus \partial_P Q_T$. We distinguish two cases:

- In the case $t_0 < T$, $(t_0, \mathbf{x}_0)$ is an interior maximum point, hence $\partial_t u_\varepsilon(t_0, \mathbf{x}_0) = 0$ and $\Delta u_\varepsilon(t_0, \mathbf{x}_0) \leq 0$. This is in contradiction with the equation $\partial u_\varepsilon = k\Delta u_\varepsilon - \varepsilon$.

- In the case $t_0 = T$, $(T, \mathbf{x}_0)$ is an interior maximum point, thus $\Delta u_\varepsilon(T, \mathbf{x}_0) \leq 0$. On the other hand, since u_ε attains its maximum at $(T, \mathbf{x}_0)$, we have

$$\partial_t u_\varepsilon(T, \mathbf{x}_0) = \lim_{s \to 0^+} \frac{u_\varepsilon(T, \mathbf{x}_0) - u_\varepsilon(T - s, \mathbf{x}_0)}{s} \geq 0\,.$$

Again these two inequalities are in contradiction with the equation $\partial_t u_\varepsilon = k\Delta u_\varepsilon - \varepsilon$.

In conclusion, u_ε attains its maximum on $\partial_P Q_T$. Since $u - \varepsilon T \le u_\varepsilon \le u$ inside Q_T we get

$$\max_{\partial_P Q_T} u \ge \max_{\partial_P Q_T} u_\varepsilon = \max_{\overline{Q_T}} u_\varepsilon \ge \max_{\overline{Q_T}} u - \varepsilon T$$

and the result follows letting $\varepsilon \to 0$. $\square$

Corollary 7.6.2. *The Dirichlet problem for the heat equation*

$$\begin{cases} u_t - k\Delta u = f\,, & \text{in } Q_T \\ u(0,\mathbf{x}) = g(\mathbf{x})\,, & \text{in } D \\ u(t,\mathbf{x}) = h(\mathbf{x})\,, & \text{in } [0,T] \times \partial D \end{cases}$$

has a unique solution.

Proof. Consider the function $v := u_1 - u_2$ and look at the equation fulfilled by v, that is the homogeneous heat equation with zero boundary and initial conditions. This leads to $v = 0$ and thus $u_1 = u_2$. The details are left as an exercise. $\square$

CHAPTER 8

LAPLACE'S EQUATION IN RECTANGULAR AND CIRCULAR DOMAINS

In this chapter we use the method of separation of variables to solve Laplace's equation on rectangular domains. We consider the domain $R = [a, b] \times [c, d]$ and we assign one boundary condition on every boundary condition. Then we conclude studying the Laplace's equation on circular domains.

8.1. Boundary condition on two opposite sides

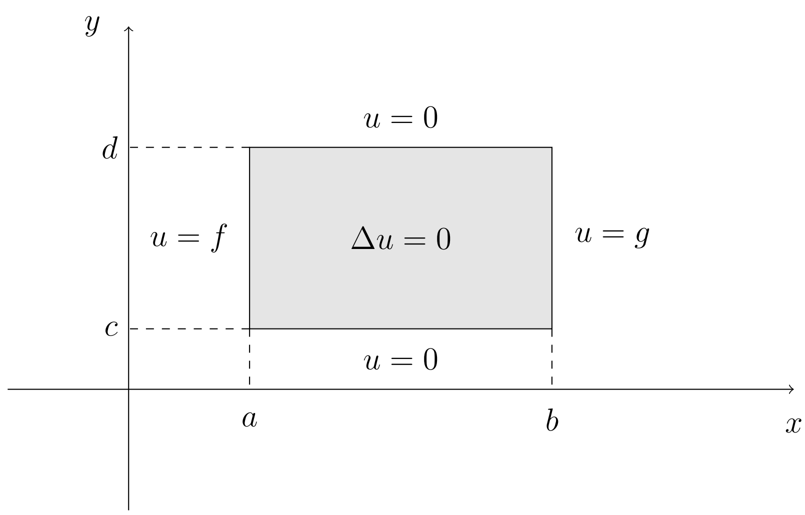

Figure 8.1: Laplace equation in a rectangular domain.

As a first example we start with the assumption that $u = 0$ on two opposite sides of the rectangle

$$\begin{cases} \Delta u = 0\,, & \text{in } R \\ u = 0\,, & \text{in } [a,b]\times\{c,d\} \\ u = f\,, & \text{in } \{a\}\times[c,d] \\ u = g\,, & \text{in } \{b\}\times[c,d]\,. \end{cases}$$

We look for a solution of the form $u(x,y) = X(x)Y(y)$. By plugging it in the equation we get $X''(x)Y(y) + Y''(y)X(x) = 0$. Dividing by $X(x)Y(y)$ we obtain

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}\,.$$

Since the function on the left only depends on x, while the one on the right only depends on y, the only possibility is that they are both constant

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda \in \mathbb{R}\,.$$

Hence, for Y we have the ODE $Y''(y) = -\lambda Y(y)$ and, by the analysis we did in Section 5.1 (and in general in Chapter 5), we know that equations of this type have three solutions depending on the sign on λ. By the condition $Y(c) = Y(d) = 0$, we deduce that λ must be positive

$$\lambda = \lambda_n = \left(\frac{n\pi}{d-c}\right)^2\,, \qquad n \in \mathbb{N}\,,\ n \geq 1\,,$$

with corresponding solution

$$Y_n(y) = a_n \sin\left(\frac{n\pi(y-c)}{d-c}\right)\,, \qquad n \in \mathbb{N}\,,\ n \geq 1\,.$$

Remark 8.1.1. If we had Neumann boundary conditions on Y, then we would have used a cosines expansion instead of a sines one and we would have started from $n = 0$ (see Section 5.2).

Concerning the ODE for X, we have $X_n''(x) = \lambda_n X_n(x)$. Since $\lambda_n > 0$ for all $n \geq 1$, the solution of our sequence of ODEs is a combination of $\sinh(\sqrt{\lambda_n}x)$ and $\cosh(\sqrt{\lambda_n}x)$. However, instead of expressing the family of all solutions to the ODE as linear combination of hyperbolic sines and cosines in x, because of our boundary conditions it is convenient to express our solution in terms of $\sinh(\sqrt{\lambda_n}(x-a))$, $\sinh(\sqrt{\lambda_n}(x-b))$ for $\lambda_n = (n\pi/(d-c))^2$. Therefore the general form of X_n is given by

$$X_n(x) = \alpha_n \sinh(\sqrt{\lambda_n}(x-a)) + \beta_n \sinh(\sqrt{\lambda_n}(x-b))\,.$$

Altogether, the expression for X_n and Y_n gives

$$u(x,y) = \sum_{n=1}^{\infty} [A_n \sinh(\sqrt{\lambda_n}(x-a)) + B_n \sinh(\sqrt{\lambda_n}(x-b))] \sin(\sqrt{\lambda_n}(y-c)) \,,$$

where we renamed the coefficients. Now the only task left is to determine the coefficients A_n and B_n. To do so we exploit the boundary conditions. Taking $x = a$, since $\sinh(0) = 0$, we get

$$u(a,y) = \sum_{n=1}^{\infty} B_n \sinh(\sqrt{\lambda_n}(a-b)) \sin(\sqrt{\lambda_n}(y-c)) = f(y) \,,$$

from which we deduce that B_n are the Fourier coefficients of g scaled by a factor $\sinh(\sqrt{\lambda_n}(a-b))$. The same reasoning applies to the other boundary condition in order to determine A_n.

8.2. Laplace's equation with Dirichlet boundary conditions in rectangular domains

In the example above, we had two opposite boundaries where u was zero. This simplified the computations and allowed us to have an expansion in sines for Y and hyperbolic sines for X.

Let us now consider the general case where we have nonzero boundary conditions

$$\begin{cases} \Delta u = 0\,, & \text{in } R \\ u = f\,, & \text{in } \{a\} \times [c,d] \\ u = g\,, & \text{in } \{b\} \times [c,d] \\ u = h\,, & \text{in } [a,b] \times \{d\} \\ u = k\,, & \text{in } [a,b] \times \{c\}\,. \end{cases} \tag{8.2.1}$$

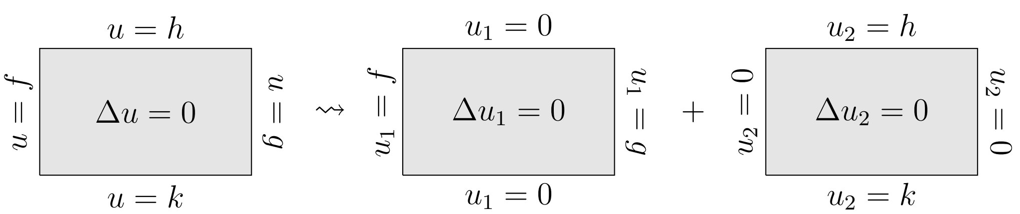

Figure 8.2: Splitting of the Laplace equation in a rectangular domain.

Then we can do the following splitting: we can write u as $u_1 + u_2$, where

$$\begin{cases} \Delta u_1 = 0\,, & \text{in } R \\ u_1 = f\,, & \text{in } \{a\}\times[c,d] \\ u_1 = g\,, & \text{in } \{b\}\times[c,d] \\ u_1 = 0\,, & \text{in } [a,b]\times\{d\} \\ u_1 = 0\,, & \text{in } [a,b]\times\{c\}\,, \end{cases} \qquad \begin{cases} \Delta u_2 = 0\,, & \text{in } R \\ u_2 = 0\,, & \text{on } \{a\}\times[c,d] \\ u_2 = 0\,, & \text{on } \{b\}\times[c,d] \\ u_2 = h\,, & \text{on } [a,b]\times\{d\} \\ u_2 = k\,, & \text{on } [a,b]\times\{c\}\,. \end{cases}$$

Hence, we saw in the previous example that u_1 is of the form

$$u_1(x,y) = \\ = \sum_{n=1}^{\infty}\left[A_n \sinh\left(\frac{n\pi}{d-c}(x-a)\right) + B_n \sinh\left(\frac{n\pi}{d-c}(x-b)\right)\right]\sin\left(\frac{n\pi}{d-c}(y-c)\right)\,.$$

Analogously, reversing the role of x and y, u_2 is given by an expression of the form

$$u_2(x,y) = \\ = \sum_{n=1}^{\infty}\left[C_n \sinh\left(\frac{n\pi}{b-a}(y-c)\right) + D_n \sinh\left(\frac{n\pi}{b-a}(y-d)\right)\right]\sin\left(\frac{n\pi}{b-a}(x-a)\right)\,.$$

Finally, note that the coefficients A_n, B_n, C_n, D_n are related to the Fourier coefficients of the boundary data.

Observe that, when we split the problem for u in two problems for u_1 and u_2, the boundary data may not be continuous anymore even if they are continuous in the original problem (consider for instance the case $f = g = h = k = 1$). This is not an issue analytically, but it becomes a problem when one wants to solve the problem numerically, since the jump in the boundary data creates numerical problems. We now describe a trick to avoid this issue.

If we want to solve (8.2.1), we can define $\overline{u} := u - P$, where P is a polynomial $P(x,y) := a_0 + a_1 x + a_2 y + a_3 xy$ for some $a_0, a_1, a_2, a_3 \in \mathbb{R}$. Note that $\overline{u}$ is still harmonic since P is harmonic, and it solves

$$\begin{cases} \Delta \overline{u} = 0\,, & \text{in } R \\ \overline{u} = \overline{f}\,, & \text{in } \{a\}\times[c,d] \\ \overline{u} = \overline{g}\,, & \text{in } \{b\}\times[c,d] \\ \overline{u} = \overline{h}\,, & \text{in } [a,b]\times\{d\} \\ \overline{u} = \overline{k}\,, & \text{in } [a,b]\times\{c\}\,, \end{cases}$$

where $\overline{f} = f - P$, $\overline{g} = g - P$, $\overline{h} = h - P$, $\overline{k} = k - P$. Now, if the boundary data for u are continuous, we can choose coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$ to ensure that $\overline{f}(a,c) = \overline{f}(a,d) = \overline{g}(b,c) = \overline{g}(b,d) = 0$ (we have four parameters to adjust four boundary conditions). In this way, if we split the problem as before in $\overline{u} = \overline{u}_1 + \overline{u}_2$, the boundary data for $\overline{u}_1$ and $\overline{u}_2$ are not discontinuous anymore.

8.3. Laplace's equation with Neumann boundary conditions in rectangular domains

Consider now the Laplace's equation in a rectangular domain with Neumann boundary conditions.

$$\begin{cases} \Delta u = 0\,, & \text{in } R \\ u_x = f\,, & \text{on } \{a\} \times [c,d] \\ u_x = g\,, & \text{on } \{b\} \times [c,d] \\ u_y = k\,, & \text{on } [a,b] \times \{d\} \\ u_y = h\,, & \text{on } [a,b] \times \{c\}\,. \end{cases}$$

Suppose that the problem satisfies the necessary condition for the existence of a

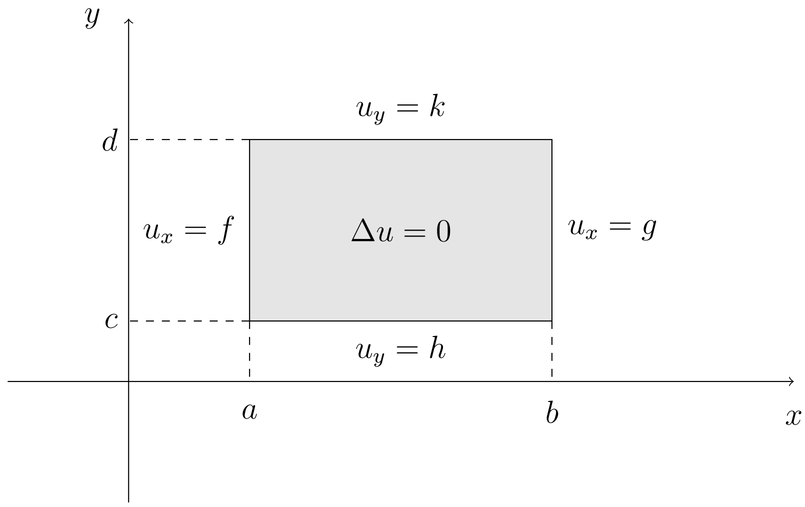

Figure 8.3: Neumann problem in a rectangular domain.

solution to the Neumann problem, namely

$$\int_c^d g - \int_c^d f + \int_a^b k - \int_a^b h = 0\,.$$

To solve the problem we need to split $u = u_1 + u_2$ in the sum of two problems as we did for the Dirichlet problem in Section 8.2. Hence u_1, u_2 satisfy

$$\begin{cases} \Delta u_1 = 0\,, & \text{in } R \\ (u_1)_x = f\,, & \text{in } \{a\} \times [c,d] \\ (u_1)_x = g\,, & \text{in } \{b\} \times [c,d] \\ (u_1)_y = 0\,, & \text{in } [a,b] \times \{d\} \\ (u_1)_y = 0\,, & \text{in } [a,b] \times \{c\}\,, \end{cases} \qquad \begin{cases} \Delta u_2 = 0\,, & \text{in } R \\ (u_2)_x = 0\,, & \text{on } \{a\} \times [c,d] \\ (u_2)_x = 0\,, & \text{on } \{b\} \times [c,d] \\ (u_2)_y = k\,, & \text{on } [a,b] \times \{d\} \\ (u_2)_y = h\,, & \text{on } [a,b] \times \{c\}\,. \end{cases}$$

Note that, by splitting the problem, the existence condition for the Neumann problem might not be satisfied anymore for u_1 and u_2. To overcome this problem, we use the trick of adding a harmonic polynomial. Consider for instance $\alpha(x^2-y^2)$ for some $\alpha \in \mathbb{R}$ and add it to u. This yields the new harmonic function $v = u+\alpha(x^2-y^2)$. If we now split $v = v_1+v_2$ as we did above for u, then the problems for v_1 and v_2 are

$$\begin{cases} \Delta u_1 = 0\,, & \text{in } R \\ (v_1)_x = f + 2\alpha a\,, & \text{in } \{a\}\times[c,d] \\ (v_1)_x = g + 2\alpha b\,, & \text{in } \{b\}\times[c,d] \\ (v_1)_y = 0\,, & \text{in } [a,b]\times\{d\} \\ (v_1)_y = 0\,, & \text{in } [a,b]\times\{c\}\,, \end{cases} \qquad \begin{cases} \Delta u_2 = 0\,, & \text{in } R \\ (v_2)_x = 0\,, & \text{in } \{a\}\times[c,d] \\ (v_2)_x = 0\,, & \text{in } \{b\}\times[c,d] \\ (v_2)_y = k - 2\alpha c\,, & \text{in } [a,b]\times\{d\} \\ (v_2)_y = h - 2\alpha d\,, & \text{in } [a,b]\times\{c\}\,. \end{cases}$$

Note that the compatibility condition for v_1 is given by

$$\int_c^d (g+2\alpha b) - \int_c^d (f+2\alpha a) = 0 \implies \alpha = \frac{1}{2(b-a)(d-c)}\int_c^d (f-g)\,.$$

Hence, with this choice of α, we can solve the problem for v_1. Now recall that, by assumption, the Neumann problem for u was solvable, that is

$$\int_c^d (g-f) + \int_a^b (k-h) = 0\,.$$

Thus α is also equal to

$$\alpha = \frac{1}{2(b-a)(d-c)}\int_c^d (f-g) = \alpha = \frac{1}{2(b-a)(d-c)}\int_a^b (k-h)\,.$$

This implies that

$$\int_a^b (k-2\alpha d) - \int_a^b (h-2\alpha c) = 0\,,$$

thus also v_2 satisfies the compatibility condition and we can solve the problem using the method of separation of variables.

8.4. Two explicit examples

Example 8.4.1. We want to solve the following Dirichlet problem on the square $R = [0,\pi]\times[0,\pi] \subset \mathbb{R}^2$

$$\begin{cases} \Delta u = 0\,, & \text{in } R \\ u(x,0) = 1 \\ u(x,\pi) = u(0,y) = u(\pi,y) = 0\,. \end{cases}$$

Since there is only one nonzero boundary condition, there is no need to split the problem as in Section 8.2. We look for a solution of the form $u(x,y) = \sum_{n\in\mathbb{N}} X_n(x)Y_n(y)$, where each term $X_n(x)Y_n(y)$ is harmonic. Hence

$$0 = \Delta(X_n(x)Y_n(y)) = X_n''(x)Y_n(y) + X_n(x)Y_n''(y)$$
$$\iff \frac{X_n''(x)}{X_n(x)} = -\frac{Y_n''(y)}{Y_n(y)} = -\lambda_n \in \mathbb{R}\,.$$

Therefore we get the system of ODEs

$$\begin{cases} X_n''(x) = -\lambda_n X_n(x)\,, & X_n(0) = X_n(\pi) = 0 \\ Y_n''(x) = \lambda_n Y_n(x)\,, \end{cases}$$

from which we deduce the solution for X_n

$$X_n(x) = A_n \sin(\sqrt{\lambda_n}x) + B_n \cos(\sqrt{\lambda_n}x)\,.$$

From $X_n(0) = 0$ we deduce $B_n = 0$ for all $n \in \mathbb{N}$, and from $X_n(\pi) = 0$ we have $\lambda_n = n^2$ for all $n \in \mathbb{N}$. Hence we get $X_n(x) = A_n \sin(nx)$ for all $n \in \mathbb{N}$.

On the other hand, the function Y_n is given by

$$Y_n(y) = C_n \sinh(ny) + D_n \sinh(n(y-\pi))\,.$$

As a result, the general solution for the problem we are considering is

$$u(x,y) = \sum_{n=1}^{\infty} \sin(nx)[C_n \sinh(ny) + D_n \sinh(n(y-\pi))]\,.$$

Remark 8.4.2. Note that the general form would have the coefficient A_n in front of the term $\sin(nx)$. However we can absorb this constant inside C_n and D_n, obtaining exactly the formula above.

From the condition $u(x,\pi) = 0$, we obtain $C_n = 0$ for all $n \in \mathbb{N}^+$. Then, from $u(x,0) = 1$, we have

$$1 = u(x,0) = \sum_{n=1}^{\infty} \sin(nx)[D_n \sin(-n\pi)] = \sum_{n=1}^{\infty} \alpha_n \sin(nx)\,,$$

where we defined $\alpha_n := D_n \sin(-n\pi)$. As usual, we multiply both sides by $\sin(mx)$ and we integrate over $[0,\pi]$, obtaining

$$\int_0^\pi \sin(mx)\,\mathrm{d}x = \sum_{n=1}^{\infty} \alpha_n \int_0^\pi \sin(mx)\sin(nx)\,\mathrm{d}x = \begin{cases} \alpha_m \pi/2\,, & \text{if } m = n \\ 0\,, & \text{if } m \neq n\,. \end{cases}$$

Thus

$$\alpha_m = \frac{2}{\pi}\int_0^\pi \sin(mx)\,\mathrm{d}x = \frac{2}{\pi}\left[-\frac{\cos(mx)}{m}\right]\Big|_0^\pi = \frac{2}{\pi}\left[\frac{1-\cos(mx)}{m}\right]$$
$$= \begin{cases} 4/(\pi m)\,, & \text{if } m \text{ is odd} \\ 0\,, & \text{if } m \text{ is even}\,. \end{cases}$$

Therefore, since $\alpha_m = D_m \sinh(-m\pi)$, we have

$$D_m = \begin{cases} \dfrac{4}{\pi m \sinh(-m\pi)}\,, & \text{if } m \text{ is odd} \\ 0\,, & \text{if } m \text{ is even}\,. \end{cases}$$

In conclusion, using that $\sinh(-m\pi) = -\sinh(m\pi)$, we get

$$u(x,y) = -\sum_{j=0}^{\infty} \frac{4\sin((2j+1)x)\sinh((2j+1)(y-\pi))}{\pi(2j+1)\sinh((2j+1)\pi)}\,.$$

Example 8.4.3. Consider now the Laplace's equation on R with Neumann boundary conditions

$$\begin{cases} \Delta u = 0\,, & \text{in } R = [0,\pi]\times[0,\pi] \\ u_y(x,\pi) = x - \pi/2 & \\ u_x(0,y) = u_x(\pi,y) = u_y(x,0) = 0\,. & \end{cases}$$

Let us verify the necessary condition to solve elliptic Neumann problems, that is $\int_{\partial R} \partial_\nu u = 0$. In our case we have

$$\int_{\partial R} \partial_\nu u = \int_0^\pi \left(x - \frac{\pi}{2}\right)\mathrm{d}x = 0 = \int_R \Delta u\,,$$

as desired. Hence we can proceed looking for a solution via the method of separation of variables

$$u(x,y) = \sum_{n\in\mathbb{N}} X_n(x)Y_n(y)\,.$$

The harmonicity condition leads to

$$\begin{cases} X_n''(x) = -\lambda_n X_n(x)\,, \quad X_n'(0) = X_n'(\pi) = 0 \\ Y_n''(x) = \lambda_n Y_n(x)\,. \end{cases}$$

Therefore we obtain $X_n(x) = \cos(nx)$ and $Y_n(y) = A_n\cosh(ny) + B_n\cosh(n(y-\pi))$ for all $n\in\mathbb{N}$. Then, the general solution is

$$u(x,y) = \sum_{n=0}^{\infty} \cos(nx)[A_n\cosh(ny) + B_n\cosh(n(y-\pi))]\,.$$

Exploiting the boundary conditions to find A_n and B_n we get

$$0 = u_y(x,0) = \sum_{n=0}^{\infty} \cos(nx) B_n n \sinh(-n\pi) \implies B_n = 0$$

and

$$x - \frac{\pi}{2} = u_y(x,\pi) = \sum_{n=0}^{\infty} A_n n \sinh(n\pi)\cos(nx) = \sum_{n=0}^{\infty} \beta_n \cos(nx)\,,$$

where $\beta_n := A_n n \sinh(n\pi)$. By a similar computation as the one in the previous example we get

$$\beta_m = \begin{cases} -\dfrac{4}{\pi m^2}\,, & \text{if } m \text{ is odd} \\ 0\,, & \text{if } m \text{ is even, } m \neq 0 \end{cases}$$

$$\implies A_m = \begin{cases} -\dfrac{4}{\pi m^3 \sinh(m\pi)}\,, & \text{if } m \text{ is odd} \\ 0\,, & \text{if } m \text{ is even, } m \neq 0\,. \end{cases}$$

This yields to the solution

$$u(x,y) = A_0 - \sum_{j=0}^{\infty} \frac{4\cos((2j+1)x)\cosh((2j+1)y)}{\pi(2j+1)^3 \sinh((2j+1)\pi)}\,.$$

Remark 8.4.4. One could also have Dirichlet conditions on some parts of the boundary and Neumann conditions on other parts of the boundary. In this case you need to choose the right bases in terms of $\sin, \cos$ and $\sinh, \cosh$.

8.5. Polar coordinates

It can be useful in applications, for example when the domain D has some radial symmetry, to express the Laplace's operator in polar coordinates. We define the *polar coordinates* (r,θ) via the relation

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta\,. \end{cases}$$

Hence, any function $u(x,y)$ can be expressed in polar coordinates via a function $w(r,\theta)$ such that $w(r,\theta) = u(x(r,\theta), y(r,\theta))$. Then the Laplacian in polar coordinates reads

$$\Delta u = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta}\,.$$

Now assume that u is a harmonic function and that w only depends on the variable r, that is $w = w(r)$, then

$$0 = \Delta u = w''(r) + \frac{1}{r} w'(r)\,.$$

By defining $v(r) := w'(r)$, we get $v'(r) = -v(r)/r$ and thus

$$\frac{v'(r)}{v(r)} = -\frac{1}{r} \iff \frac{\mathrm{d}}{\mathrm{d}r} \log|v(r)| = -\frac{\mathrm{d}}{\mathrm{d}r} \log r \iff \log|v(r)| = -\log(r) + c\,,$$

for some constant $c \in \mathbb{R}$. Hence we obtain that $w'(r) = v(r) = e^c/r$ if $v(r) > 0$ and $w'(r) = v(r) = -e^c/r$ if $v(r) < 0$. Integrating with respect to r we get

$$w(r) = \pm \int_1^r \frac{e^c}{s} \,\mathrm{d}s \pm w(1) = c_1 \log(r) + c_2\,,$$

with $c_1 = e^c > 0$ if $v(r) > 0$ and $c_1 = -e^c < 0$ if $v(r) < 0$.

Then $w(r) = c_1 \log(r) + c_2$ is a solution of Laplace's equation for $r > 0$. Since $r = \sqrt{x^2 + y^2}$, this proves that $u(x, y) = w(r) = c_1 \log(\sqrt{x^2 + y^2}) + c_2 = c_1 \log(x^2 + y^2)/2 + c_2$ is harmonic on $\mathbb{R}^2 \setminus \{(0, 0)\}$ for any $c_1, c_2 \in \mathbb{R}$.

8.6. Laplace's equation in circular domains

We now consider Laplace's equation on circular domains $D = B_a = \{0 \leq r \leq a, \theta \in [0, 2\pi]\}$. For this problem we use the expression of the Laplacian in polar coordinates

$$0 = \Delta u = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta},$$

where $u(x(r, \theta), y(r, \theta)) = u(r \cos\theta, r \sin\theta) = w(r, \theta)$. We look for separated solutions of the form

$$w(r, \theta) = R(r)\Theta(\theta)$$

and obtain

$$\begin{aligned} 0 &= R''(r)\Theta(\theta) + \frac{1}{r} R'(r)\Theta(\theta) + \frac{1}{r^2} \Theta''(\theta)R(r) \\ \implies & \frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda. \end{aligned}$$

Hence we have the ODEs system

$$\begin{cases} r^2 R''(r) + rR'(r) = \lambda R(r) \\ \Theta''(\theta) = -\lambda \Theta(\theta)\,, \qquad \Theta(0) = \Theta(2\pi), \Theta'(0) = \Theta'(2\pi)\,. \end{cases}$$

Note that the conditions $\Theta(0) = \Theta(2\pi), \Theta'(0) = \Theta'(2\pi)$ come from the fact that we want u to be a classical solution inside D, so it should be at least C^2. Hence we impose that Θ and Θ' are periodic in $[0, 2\pi]$. Observe that, since $\Theta'' = -\lambda\Theta$ automatically also Θ'' is periodic. The solution for the second ODE is

$$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)\,, \qquad n \in \mathbb{N}\,.$$

For the first equation one can check that

$$R_n(r) = \begin{cases} C_0 + D_0 \log r\,, & \text{for } n = 0 \\ C_n r^n + D_n r^{-n}\,, & \text{for } n \neq 0 \end{cases}$$

gives the two parameter family of solutions. However the functions r^{-n} and $\log r$ are singular at 0 inside the domain D, so we discard them. Thus the general solution is given by

$$w(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]\,.$$

Remark 8.6.1. The same method as above can be applied to domains that are discs, circles, rings or sectors of a circle/ring. However, in the cases where the domain is only a sector of a disc or a ring, then Θ is not necessarily periodic anymore. Moreover, in cases where the origin is not in the domain, we do not have to discard the terms with r^{-n} and $\log r$.

Example 8.6.2. Let $B_1 = \{x^2 + y^2 \leq 1\}$ be the unit disc in $\mathbb{R}^2$. We want to solve the following Dirichlet problem

$$\begin{cases} \Delta u = 0\,, & \text{in } B_1 \\ u = y^2\,, & \text{in } \partial B_1\,. \end{cases}$$

Using polar coordinates and defining $w(r, \theta) = u(r \cos\theta, r \sin\theta)$, we can rewrite the problem as

$$\begin{cases} w_{rr} + \dfrac{1}{r} w_r + \dfrac{1}{r^2} w_{\theta\theta} = 0\,, & (r, \theta) \in (0, 1) \times (0, 2\pi) \\ w(1, \theta) = \sin^2\theta = \dfrac{1}{2} - \dfrac{1}{2}\cos(2\theta)\,. & \end{cases}$$

As seen before, we then get that the general solution has the form

$$w(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]\,.$$

Imposing the boundary condition we have

$$\frac{1}{2} - \frac{1}{2}\cos(2\theta) = w(1,\theta) = C_0 + \sum_{n=1}^{\infty}[A_n\cos(n\theta) + B_n\sin(n\theta)]\,,$$

from which we deduce that $C_0 = 1/2$, $A_2 = -1/2$ and all the other coefficients are zero. Thus, the final solution is

$$w(r,\theta) = \frac{1}{2} - \frac{1}{2}r^2\cos(2\theta) \implies u(x,y) = \frac{1}{2}(1 - x^2 - y^2)\,.$$

Example 8.6.3. Let us consider the problem

$$\begin{cases} \Delta u = 0\,, & \text{on } D = \{(x,y) \in \mathbb{R}^2 \ :\ 1 \le \sqrt{x^2+y^2} \le 2\} \\ u(x,y) = 3x/2\,, & \text{on } \{x^2+y^2 = 2\} \\ u(x,y) = y\,, & \text{on } \{x^2+y^2 = 1\}\,. \end{cases}$$

In polar coordinates this problem reads as

$$\begin{cases} w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} = 0\,, \quad \text{in } D = \{r \in [1,2], \theta \in [0,2\pi)\} \\ w(2,\theta) = 3\cos\theta \\ w(1,\theta) = \sin\theta\,. \end{cases}$$

If we write $w(r,\theta) = R(r)\Theta(\theta)$, the ODEs for R and Θ are the same as before, but now the boundary conditions for R have changed. The general solution is

$$w(r,\theta) = E + F\log r + \\ + \sum_{n=1}^{\infty}[A_n r^n\cos(n\theta) + B_n r^n\sin(n\theta) + C_n r^{-n}\cos(n\theta) + D_n r^{-n}\sin(n\theta)]\,.$$

Using the boundary condition $w(1,\theta) = \sin\theta$, we have

$$\sin\theta = w(1,\theta) = E + \sum_{n=1}^{\infty}[(A_n + C_n)\cos(n\theta) + (B_n + D_n)\sin(n\theta)]\,,$$

which implies $E = 0$, $B_1 + D_1 = 1$, $B_n + D_n = 0$ for all $n \ge 2$ and $A_n + C_n = 0$ for all $n \ge 1$. By the condition $w(2,\theta) = 3\cos\theta$, we have

$$3\cos\theta = w(2,\theta) = E + F\log(2) + \\ + \sum_{n=1}^{\infty}[(2^n A_n + 2^{-n}C_n)\cos(n\theta) + (2^n B_n + 2^{-n}D_n)\sin(n\theta)]\,.$$

This implies that $E+F\log(2)=0$, $2^nB_n+2^{-n}D_n=0$ for all $n\geq 1$, $2A_1+C_1/2=3$, $2^nA_n+2^{-n}C_n=0$ for all $n\geq 2$. Combining all these information we get

$$
\begin{aligned}
&E=0, E+F\log(2)=0 \implies E=F=0\\
&A_1+C_1=0, 2A_1+\frac{1}{2}C_1=3 \implies A_1=2, C_1=-2\\
&B_1+D_1=1, 2B_1+\frac{1}{2}D_1=0 \implies B_1=-\frac{1}{3}, D_1=\frac{4}{3},
\end{aligned}
$$

and for $n\geq 2$

$$
\begin{aligned}
&A_n+B_n=0, 2^nA_n+2^{-n}C_n=0 \implies A_n=C_n=0\\
&B_n+D_n=0, 2^nB_n+2^{-n}D_n=0 \implies B_n=D_n=0\,.
\end{aligned}
$$

This proves that

$$
w(r,\theta)=2r\cos\theta-\frac{1}{3}r\sin\theta-2r^{-1}\cos\theta+\frac{4}{3}r^{-1}\sin\theta\,.
$$

Example 8.6.4. Let us now consider Laplace's equation on an annular sector of angle $\gamma\in(0,2\pi)$ and radii 1 and 2, i.e., $D=\{(r,\theta)\ :\ r\in(1,2),\theta\in(0,\gamma)\}$. To solve such problem we rely on the formula for the Laplacian in polar coordinates and on the method of separation of variables. Assume that w is prescribed on ∂D and that $w(r,0)=w(r,\gamma)=0$ for all $r\in(1,2)$. If we look for solutions of the form $w(r,\theta)=R(r)\Theta(\theta)$, to enforce these boundary conditions we impose $\Theta(0)=\Theta(\gamma)=0$. Hence we have

$$
\Theta_n(\theta)=A_n\sin\left(\frac{n\pi}{\gamma}\theta\right)\,.
$$

Then, the ODE for R_n becomes

$$
r^2R_n''(r)+rR_n'(r)-\lambda_nR_n(r)=0\,,
$$

where $\lambda_n=(n\pi/\gamma)^2$. Looking for solutions of the form r^α, we get

$$
0=\alpha(\alpha-1)+\alpha-\lambda_n=\alpha^2-\lambda_n \implies \alpha=\pm\sqrt{\lambda_n}\,.
$$

Hence $R_n(r)=C_nr^{n\pi/\gamma}+D_nr^{-n\pi/\gamma}$ and the general solution in this case is given by

$$
w(r,\theta)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{\gamma}\theta\right)r^{n\pi/\gamma}+B_n\sin\left(\frac{n\pi}{\gamma}\theta\right)r^{-n\pi/\gamma}
$$

and the coefficients A_n and B_n are found expanding the boundary conditions $w(1,\theta)$ and $w(2,\theta)$ over the interval $\theta\in[0,\gamma]$ using the Fourier basis $\{\sin(n\pi\theta/\gamma)\}$.

Remark 8.6.5. If the sector is $D = \{(r, \theta) \ : \ r \in [0, 2), \theta \in (0, \gamma)\}$ with boundary conditions $w(r, 0) = w(r, \gamma) = 0$ for all $r \in (0, 2)$, then the general solution is of the form

$$w(r, \theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{n\pi/\gamma}\,,$$

since the negative powers of r are singular at the origin and should be discarded.

8.7. A "real life" example

Consider a pair of infinite, grounded conducting sheets separated at distance d. Suppose that there is a conductor connecting the two metal sheets held at position $U_0 \sin(2\pi x/d)$. We want to understand what is the potential between the plates.

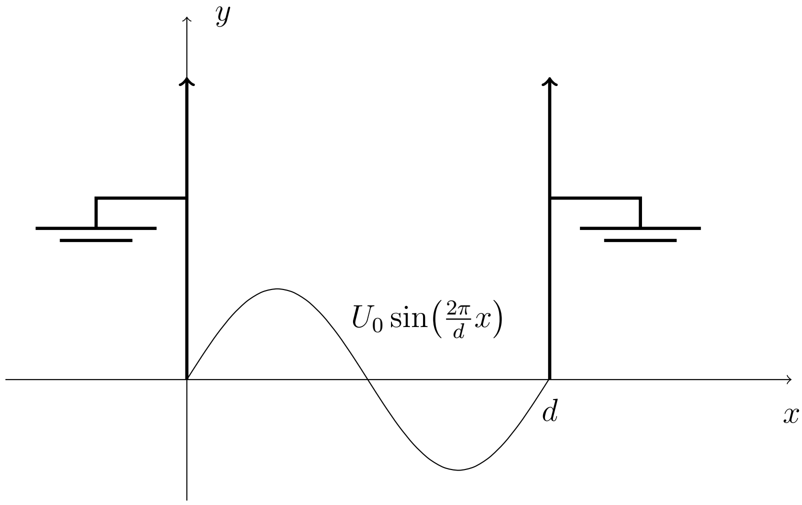

Figure 8.4: Configuration of the conductor between two plates.

We know that the electric potential satisfies Laplace's equation in the region between plates (since there is no charge in there). Therefore we want to solve the following Dirichlet problem

$$\begin{cases} \Delta u = 0\,, & \text{in } (0, d) \times \mathbb{R}^+ \\ u(x, 0) = U_0 \sin(2\pi x/d)\,, & \\ u(0, y) = u(d, y) = 0\,. & \end{cases}$$

Note that there is an additional implicit boundary condition: we would like the potential to go to zero in the "open" spatial direction, that in formulas translates to

$$\lim_{y \to \infty} u(x, y) = 0\,. \tag{8.7.1}$$

Let us suppose that $u(x,y) = \sum_{n\in\mathbb{N}} X_n(x)Y_n(y)$, with $X_n(x)Y_n(y)$ harmonic for all $n \in \mathbb{N}$. This leads to the ODEs

$$\begin{cases} X_n''(x) = -\lambda_n X_n(x)\,, \quad X_n(0) = X_n(d) = 0 \\ Y_n''(y) = \lambda_n Y_n(y)\,. \end{cases}$$

The solution to the first ODE is

$$X_n(x) = A_n \sin(\sqrt{\lambda_n} x)\,, \qquad \text{with } \lambda_n = \left(\frac{n\pi}{d}\right)^2 \text{ for all } n \in \mathbb{N}\,.$$

On the other hand the solution to the second ODE is

$$Y_n(y) = C_n \sinh(\sqrt{\lambda_n} y) + D_n \cosh(\sqrt{\lambda_n} y)\,.$$

By condition (8.7.1), we deduce

$$\lim_{y\to\infty} \left(C_n \frac{e^{\sqrt{\lambda_n} y} - e^{-\sqrt{\lambda_n} y}}{2} + D_n \frac{e^{\sqrt{\lambda_n} y} + e^{-\sqrt{\lambda_n} y}}{2} \right) =$$

$$= \lim_{y\to\infty} \left(\frac{C_n + D_n}{2}\right) e^{\sqrt{\lambda_n} y} + \left(\frac{D_n - C_n}{2}\right) e^{-\sqrt{\lambda_n} y} = 0\,.$$

Therefore we have that $C_n + D_n = 0$ for all $n \in \mathbb{N}$. Hence

$$Y_n(y) = D_n(\cosh(\sqrt{\lambda_n} y) - \sinh(\sqrt{\lambda_n} y)) = D^n e^{-\sqrt{\lambda_n} y}$$

and the general solution is given by

$$u(x,y) = \sum_{n\in\mathbb{N}} A_n \sin\left(\frac{n\pi}{d} x\right) e^{-n\pi y/d}\,.$$

By the condition $u(x,0) = U_0 \sin(2\pi x/d)$, we deduce that $A_n = 0$ for all $n \neq 2$ and the final solution is

$$u(x,y) = U_0 \sin\left(\frac{2\pi}{d} x\right) e^{-2\pi y/d}\,.$$

Bibliography

[Pin05] Yehuda Pinchover. *Introduction to partial differential equations.* Cambridge University Press, Cambridge, 2005.