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The axiom of determinacy and Lebesgue measurability

Bachelor Thesis

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Abstract

Building on Donald A. Martin's work "A simple proof that determinacy implies Lebesgue measurability", this thesis provides a detailed proof that the axiom of determinacy implies that all sets of real numbers are Lebesgue measurable. To enhance accessibility, omitted steps from Martin's paper are filled in, and basic concepts from measure theory and game theory are systematically introduced. Finally, this thesis extends Martin's results to the context of multidimensional Lebesgue measure.

Preface

Historical context

The debate around the existence of non-measurable sets has lead to many fascinating results. In 1905 Giuseppe Vitali proofed the existence of a non-measurable set utilizing the axiom of choice [11]. However, the choice function used in Vitali's proof is not measurable and cannot be defined in a natural way, leading to historical controversy. Henri Lebesgue himself expressed skepticism that a non-measurable set could ever be defined uniquely [6, ch. 2.3].

Substantial progress in this debate was made in 1970 when Robert M. Solovay constructed a model of set theory in which every set of real numbers is Lebesgue measurable, providing rigorous mathematical support for Lebesgue's concerns [9]. A less constructive alternative to Solovay's approach is to replace the axiom of choice with the axiom of determinacy, introduced by Jan Mycielski and Hugo Steinhaus in 1962 [7]. Indeed the axiom of determinacy implies that every set of real numbers is measurable; a result first shown by Jan Mycielski and Stanisław Świerczkowski in 1964 [8].

Purpose of this text

The objective of this bachelor thesis is to thoroughly investigate and elaborate on the proof presented in the article "A simple proof that determinacy implies Lebesgue measurability" by Donald A. Martin [4] and to develop a generalization of Martin's proof, extending it to the multidimensional Lebesgue measure. The aim is that a mathematics bachelor student with rudimentary knowledge of set theory and measure theory can comprehend these proofs without requiring additional research.

Structure

This thesis is divided into three chapters. The initial section of the first chapter provides an overview of relevant measure-theoretic results and definitions. Following this, the axiom of determinacy is introduced along with the notation for games that will be used throughout the thesis.

The second chapter shows that the axiom of determinacy implies that all sets are measurable. To achieve this rigorously, auxiliary tools are constructed and necessary lemmas are proven.

The third and final chapter summarizes the strategy used to prove that the axiom of determinacy implies that all sets are measurable. Additionally, it discusses an approach to generalize this result to subsets of $\mathbb{R}^m$. The two key lemmas of this generalization are proven in Appendix A and Appendix B, respectively.

Notation

This section clarifies the notation used throughout the thesis.

The set of real numbers is denoted by $\mathbb{R}$, and intervals are represented using angled and closed brackets (i.e., $(a, b]$ denotes the interval from a to b, excluding a but including b).

The smallest nonempty limit ordinal, corresponding to the natural numbers, is denoted by $\omega = \{0, 1, 2, \ldots\}$. Thus, for all $n \in \omega$ we have $n = \{0, 1, \ldots, n-1\}$, which allows for more compact notation in certain contexts.

For two sets A and B, the set of all functions from A to B is denoted by ${}^{A}B$ (e.g., ${}^{\omega}\omega$ represents the set of all infinite sequences of natural numbers). For $f \in {}^{A}B$ and subsets $A' \subseteq A$, the function f restricted to A' is denoted by $f|_{A'}$, and the image of A' under f is denoted by $f(A')$.

Sequences are denoted using angled brackets. Specifically, given some $n \in \omega$, a set A, and elements $a_0, \ldots, a_{n-1} \in A$, the function

$$n \to A, k \mapsto a_k$$

is denoted by $\langle a_0, \ldots, a_{n-1} \rangle$ or by $\langle a_k : k \in n \rangle$. Similarly, given elements $a_k \in A$ for all $k \in \omega$, the function

$$\omega \to A, k \mapsto a_k$$

is denoted by $\langle a_0, a_1, a_2, \ldots \rangle$ or by $\langle a_k : k \in \omega \rangle$.

Finally, given two finite sequences $\langle a_0, \ldots, a_{n-1} \rangle$ and $\langle b_0, \ldots, b_{m-1} \rangle$, their concatenation is defined as follows:

$$\langle a_0, \ldots, a_{n-1} \rangle^\frown \langle b_0, \ldots, b_{m-1} \rangle := \langle a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1} \rangle.$$

Additional notation for more complex topics will be introduced and discussed in detail as needed.

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Contents

1 Intruduction

1.1 Lebesgue measure

This section presents the notations and definitions of the measure-theoretic concepts used in this thesis along with some useful facts. Most results from this section won't be proven here, but references for proofs will be provided. For a more detailed discussion about measure theory, see [1] or [2].

First, we define the (outer) Lebesgue measure on $\mathbb{R}$:

Definition 1.1. *For $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, we define*

$$\mu\big((a,b)\big) = \mu\big([a,b)\big) = \mu\big((a,b]\big) = \mu\big([a,b]\big) = b - a,$$

where the convention $+\infty - a = b - (-\infty) = +\infty$ is used.
Given an arbitrary set $A \subseteq \mathbb{R}$, the outer measure of A is defined as

$$\mu^*(A) := \inf\left\{\sum_{k=0}^{\infty} \mu(I_k) : \text{each } I_k \text{ is an interval and } A \subseteq \bigcup_{k=0}^{\infty} I_k\right\}.$$

Further A is called measurable if for all sets $B \subseteq \mathbb{R}$ it holds

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A).$$

For measurable sets $A \subseteq \mathbb{R}$ we call $\mu(A) := \mu^*(A)$ the measure of A.

Remark 1.2. *The abuse of notation might lead to confusion, as $\mu(I)$ has already been defined for intervals $I \subseteq \mathbb{R} \cup \{\pm\infty\}$. However, every interval $I \subseteq \mathbb{R}$ is measurable, and the two definitions of $\mu(I)$ coincide. For a proof, see [1, Lemma 9.5].*

The definition of the outer measure raises the question of whether an inner measure exists too. Indeed, an inner measure can be defined, but one has to be somewhat careful; the inner measure is not simply some supremum of the measure of inscribing intervals.

Definition 1.3. *If $A \subseteq [-n, n]$ for some $n \in \omega$, the inner measure of A is defined as*

$$\mu_*(A) := \mu\big([-n,n]\big) - \mu^*\big([-n,n] \setminus A\big).$$

It generally holds that $\mu_*(A) \leq \mu^*(A)$ and further $\mu_*(A) = \mu^*(A)$ if and only if A is measurable [1, Corollary 16.4]. Therefore the inner measure can be used as a tool to check whether a set is measurable.

To estimate the inner and outer measure of sets we will rely on the following fact.

Fact 1.4. *For any set $B \subseteq [-1, 1]$ and measurable sets $A, C \subseteq \mathbb{R}$ with $A \subseteq B \subseteq C$, it holds that*

$$\mu(A) \leq \mu_*(B) \leq \mu^*(B) \leq \mu(C).$$

Proof. The last inequality, $\mu^*(B) \leq \mu(C)$, follows directly from the definition of the outer measure. The inequality $\mu_*(B) \leq \mu^*(B)$ was discussed above.

For the first inequality, $\mu(A) \leq \mu_*(B)$, note that $[-1,1] \setminus B \subseteq [-1,1] \setminus A$, and thus

$$\mu^*\big([-1,1] \setminus B\big) \leq \mu^*\big([-1,1] \setminus A\big).$$

Using this inequality and the fact that $A \subseteq [-1,1]$ is measurable, we conclude

$$\mu(A) + \mu^*\big([-1,1] \setminus B\big) \leq \mu^*\big([-1,1] \cap A\big) + \mu^*\big([-1,1] \setminus A\big) = \mu([-1,1]),$$

and therefore,

$$\mu(A) \leq \mu([-1,1]) - \mu^*\big([-1,1] \setminus B\big) = \mu_*(B).$$

□

Finally, the following fact about the measure of unions and intersections will be needed. This is a part of [2, Theorem 1.2], where a proof can be found.

Fact 1.5. *For any sequence $\langle A_k : k \in \omega \rangle$ of measurable sets the following statements hold:*

(i) The union $\bigcup_{k\in\omega} A_k$ *and intersection* $\bigcap_{k\in\omega} A_k$ *are measurable.*

(ii) If $A_k \subseteq A_{k+1}$ *for all* $k \in \omega$ *then* $\mu\Big(\bigcup_{k\in\omega} A_k\Big) = \lim_{k\to\infty} \mu(A_k)$.

(iii) If $A_k \supseteq A_{k+1}$ *for all* $k \in \omega$ *and* $\mu(A_0) < \infty$ *then* $\mu\Big(\bigcap_{k\in\omega} A_k\Big) = \lim_{k\to\infty} \mu(A_k)$.

1.2 Games and the axiom of determinacy

Having discussed some basic measure-theoretic results, we now turn to an introduction to games. The axiom of determinacy concerns the winning strategies of games, necessitating the introduction of specific notations and definitions. This section is based on [3, ch. 15&16], where an illustrative discussion of this subject is provided.

Definition 1.6. *The infinite two-player game with perfect information called $\mathcal{G}_A$, for some fixed $A \subseteq {}^{\omega}\omega$, is a turn-based game where two players alternate playing natural numbers:*

$$\begin{array}{rllllllll} \mathrm{I}: & a_0 & & a_1 & & a_2 & & \dots & \\ \mathrm{II}: & & b_0 & & b_1 & & b_2 & & \dots \end{array}$$

The starting player is denoted as I *and the other player as* II*, with each player having complete knowledge of all previous moves. The resulting sequence $z := \langle a_0, b_0, a_1, b_1, \dots \rangle \in {}^{\omega}\omega$ is called a* ***play of*** $\mathcal{G}_A$*. Player* I *wins the game if $z \in A$, and player* II *wins if $z \notin A$. There are no draws.*

We want to discuss the concept of strategies in games. Intuitively, a strategy dictates how a player should respond to the moves of their opponent.

Definition 1.7. *A strategy for player* I *is a map*

$$\sigma : \bigcup_{n \in \omega} {}^{2n}\omega \to \omega,$$

and a strategy for player II *is a map*

$$\tau : \bigcup_{n \in \omega} {}^{2n+1}\omega \to \omega.$$

Thus, a strategy σ for player I assigns to every even-length sequence of natural numbers another natural number, representing the next move by player I. Conversely, a strategy τ for player II assigns to every odd-length sequence of natural numbers another natural number, representing the next move by player II.

Definition 1.8. *To express that player* I *follows a strategy σ, we say that a play $z = \langle a_0, b_0, a_1, b_1, \dots \rangle$ of $\mathcal{G}_A$* ***is consistent with*** *σ if*

$$\forall n \in \omega : a_n = \sigma(\langle a_0, b_0, a_1, b_1, \dots, a_{n-1}, b_{n-1} \rangle).$$

Similarly, to express that player II *follows a strategy τ, we say that z* ***is consistent with*** *τ if*

$$\forall n \in \omega : b_n = \tau(\langle a_0, b_0, a_1, \dots, b_{n-1}, a_n \rangle).$$

Now we define a notion to determine how player I would react to a sequence of moves by player II.

Definition 1.9. *For $\alpha \in \omega \cup \{\omega\}$ and a (finite or infinite) sequence $b = \langle b_n : n \in \alpha \rangle \in {}^{\alpha}2$ of moves by player* II*, we define*

$$\sigma * b := \begin{cases} \langle a_0, b_0, a_1, b_1, \dots \rangle \in {}^{\omega}\omega & \text{if } \alpha = \omega, \\ \langle a_0, b_0, a_1, b_1, \dots, a_\alpha \rangle \in {}^{2\alpha+1}\omega & \text{if } \alpha \in \omega. \end{cases}$$

where $a_n := \sigma(\langle a_0, b_0, \dots, a_{n-1}, b_{n-1} \rangle)$ for all $n \in \omega$ if $\alpha = \omega$ or for all $n \leq \alpha$ if $\alpha \in \omega$, respectively.

At this point we introduce the concept of a winning strategy and the determinacy of games.

Definition 1.10. *A strategy ρ for player* I *(or player* II*) is called a winning strategy if every play of $\mathcal{G}_A$ consistent with ρ is a win for player* I *(or player* II*).*

Definition 1.11. *A game $\mathcal{G}_A$ is called determined if there exists a winning strategy for player* I *or player* II.

Finally, we are able to introduce the axiom of determinacy:

Definition 1.12. *The axiom of determinacy (*AD*) states that for all $A \subseteq {}^{\omega}\omega$, the game $\mathcal{G}_A$ is determined.*

2 Proving measurability through determinacy

In this chapter, we will rigorously prove that the axiom of determinacy implies that all sets of reals are Lebesgue measurable. The proofs presented here are derived from Martin's work in [4], which itself builds on the ideas proposed by Martin [5] and Vervoort [10].

2.1 Binary representation of real numbers

We first prove that all subsets of $[0,1]$ are measurable and then generalize it to $\mathbb{R}$. However, as we are working with games, it is easier to first identify $[0,1]$ with the set of all $\{0,1\}$ sequences of length ω. This can be achieved by identifying such a $\{0,1\}$ sequence as the decimal places of a binary number in $[0,1]$. This function can rigorously be defined as

$$g : {}^{\omega}2 \to [0,1], \quad x \mapsto \sum_{n=0}^{\infty} x(n) 2^{-(n+1)}.$$

The following lemma clarifies how the initial segment of a sequence determines its image under g.

Lemma 2.1. *For all $n \in \omega$ and for all $p \in {}^{n}2$ it holds*

$$g\big(\{x \in {}^{\omega}2 : x|_n = p\}\big) = \Big[\sum_{k=0}^{n-1} p(k) 2^{-(k+1)} \ , \ 2^{-n} + \sum_{k=0}^{n-1} p(k) 2^{-(k+1)}\Big]. \tag{1}$$

Proof. Let $n \in \omega$ and $p \in {}^{n}2$ be arbitrary. For all $x \in {}^{\omega}2$ with $x|_n = p$ it holds

$$0 \leq \sum_{k=0}^{\infty} x(k) 2^{-(k+1)} - \sum_{k=0}^{n-1} p(k) 2^{-(k+1)} = \sum_{k=n}^{\infty} x(k) 2^{-(k+1)} \leq \sum_{k=n}^{\infty} 2^{-(k+1)} = 2^{-n},$$

or equivalently $g(x) \in \Big[\sum_{k=0}^{n-1} p(k) 2^{-(k+1)}, 2^{-n} + \sum_{k=0}^{n-1} p(k) 2^{-(k+1)}\Big]$. Thus, we have proved one inclusion of equation (1).

To show the other inclusion, let $y \in \Big[\sum_{k=0}^{n-1} p(k) 2^{-(k+1)}, 2^{-n} + \sum_{k=0}^{n-1} p(k) 2^{-(k+1)}\Big]$ be arbitrary. We recursively define

$$y_k = \begin{cases} p(k) & \text{if } k < n, \\ 0 & \text{if } k \geq n \text{ and } y - \sum_{j=0}^{k-1} y_j 2^{-(j+1)} < 2^{-(k+1)}, \\ 1 & \text{else.} \end{cases}$$

By induction, one can show that

$$0 \leq y - \sum_{k=0}^{\ell} y_k 2^{-(k+1)} \leq 2^{-(\ell+1)}$$

for all $\ell \geq n$. Taking the limit as $\ell \to \infty$, we get

$$g(\langle y_k : k \in \omega \rangle) = \lim_{\ell \to \infty} \sum_{k=0}^{\ell} y_k 2^{-(k+1)} = y.$$

□

Note that for $n = 0$ this lemma implies that $g({}^\omega 2) = [0,1]$, thus g is a well-defined surjection. It is natural to ask whether g is even bijective, which would mean that each real number in $[0,1]$ has a unique binary representation. This is not entirely the case, as, for instance,

$$g(\langle 1,0,0,0,\dots\rangle) = \frac{1}{2} = g(\langle 0,1,1,1,\dots\rangle).$$

However, the next lemma establishes a weaker form of injectivity.

Lemma 2.2. *For each $t \in [0,1]$, it holds $|g^{-1}(\{t\})| \leq 2$.*

Proof. Assume by contradiction that there exists $t \in [0,1]$ such that $|g^{-1}(\{t\})| > 2$. Choose distinct elements $x, x', x'' \in g^{-1}(\{t\})$. As $x \neq x'$, there exists a minimal $n \in \omega$ with $x(n) \neq x'(n)$. Without loss of generality, assume that $x(n) = 1$ and $x'(n) = 0$. It holds

$$\begin{aligned}
0 = g(x') - g(x) &= \sum_{k=0}^{\infty} x'(k)2^{-(k+1)} - \sum_{k=0}^{\infty} x(k)2^{-(k+1)} \\
&= \sum_{k=n}^{\infty} \big(x'(k) - x(k)\big)2^{-(k+1)} \\
&= \big(x'(n) - x(n)\big)2^{-(n+1)} + \sum_{k=1}^{\infty} \big(x'(k+n) - x(k+n)\big)2^{-(k+n+1)} \\
&= 2^{-(n+1)}\Big(-1 + \sum_{k=1}^{\infty} \big(x'(k+n) - x(k+n)\big)2^{-k}\Big).
\end{aligned}$$

This is equivalent to

$$\sum_{k=1}^{\infty} \big(x'(k+n) - x(k+n)\big)2^{-k} = 1,$$

which, as $x'(k+n) - x(k+n) \leq 1$, is exactly the case if $x'(k+n) - x(k+n) = 1$ for all $k \geq 1$. Thus we have $x(k) = 0$ and $x'(k) = 1$ for all $k > n$. Since by assumption $x(n) = 1$ it follows $n = \max\{k \in \omega : x(k) = 1\}$.

Now consider the minimal $m \in \omega$ such that $x(m) \neq x''(m)$. With the same argumentation as above and the fact that x cannot be 1 at infinitely many points, we get $x(k) = 0$ and $x''(k) = 1$ for all $k > m$ and further $m = \max\{k \in \omega : x(k) = 1\} = n$. Thus, we conclude that $x'(k) = x''(k)$ for all $k \in \omega$, and consequently $x' = x''$. This contradicts the assumption that x' and x'' are distinct. ☐

We can use this result to transfer part (iii) of Fact 1.5 to $\{0,1\}$ sequences.

Corollary 2.3. *For any sequence $\langle X_n : n \in \omega\rangle$ with ${}^\omega 2 \supseteq X_0 \supseteq X_1 \supseteq \dots$, it holds*

$$g\Big(\bigcap_{n\in\omega} X_n\Big) = \bigcap_{n\in\omega} g(X_n).$$

If, further, $g(X_n)$ is measurable for all $n \in \omega$, then $g\Big(\bigcap_{n\in\omega} X_n\Big)$ is measurable with

$$\mu\Big(g\Big(\bigcap_{n\in\omega} X_n\Big)\Big) = \lim_{n\to\infty} \mu(g(X_n)).$$

Proof. Let $y \in [0,1]$ be arbitrary. Since g is surjective and by Lemma 2.2, there exist (not necessarily distinct) $x_0, x_1 \in {}^{\omega}2$ such that $\{x_0, x_1\} = g^{-1}(\{y\})$. We show the following equivalences to prove the first part. Hereby the only implication which doesn't follow immediately is (3) $\Rightarrow$ (4).

$$
\begin{aligned}
y \in \bigcap_{n\in\omega} g(X_n) &\Leftrightarrow \forall n \in \omega : y \in g(X_n) && (2)\\
&\Leftrightarrow \forall n \in \omega : \big(x_0 \in X_n \vee x_1 \in X_n\big) && (3)\\
&\Leftrightarrow \exists i \in \{0,1\}\, \forall n \in \omega : x_i \in X_n && (4)\\
&\Leftrightarrow y \in g\Big(\bigcap_{n\in\omega} X_n\Big) && (5)
\end{aligned}
$$

We prove the implication (3) $\Rightarrow$ (4) by contraposition: If (4) doesn't hold, fix $i \in \{0,1\}$. We can find $n_i \in \omega$ such that $x_i \notin X_{n_i}$. But since $X_0 \supseteq X_1 \supseteq \ldots$, we get $x_i \notin X_m$ for all $m \geq n_i$. In particular, for $n \geq \max\{n_0, n_1\}$, we have $x_0 \notin X_n$ and $x_1 \notin X_n$, and therefore (3) does not hold.

Now assume that $g(X_n)$ is measurable for all $n \in \omega$. Then $g\Big(\bigcap_{n\in\omega} X_n\Big) = \bigcap_{n\in\omega} g(X_n)$ is measurable by part (i) of Fact 1.5. Finally, note that $g(X_k) \supseteq g(X_{k+1})$ for all $k \in \omega$ and $\mu(g(X_0)) \leq \mu([0,1]) = 1 < \infty$. Consequently, by part (iii) of Fact 1.5, we get

$$\mu\Big(g\Big(\bigcap_{n\in\omega} X_n\Big)\Big) = \mu\Big(\bigcap_{n\in\omega} g(X_n)\Big) = \lim_{n\to\infty} \mu(g(X_n)).$$

□

2.2 Game setup

Definition 1.12 suggests we need to work with games and the fact that they are determined to prove that the axiom of determinacy implies the measurability of all subsets of $[0,1]$.

Since the function g introduced in the previous section is surjective (as explained after Lemma 2.1), it suffices to show that for all $X \subseteq {}^{\omega}2$ the set $g(X) \subseteq [0,1]$ is measurable.

With this established, we now introduce the game to be used for the remainder of the thesis. Fix some $X \subseteq {}^{\omega}2$ as well as some $v \in (0,1]$ and consider the following game $\mathcal{G}_{v,X}$. Player I selects pairs of rational numbers $h_n \in {}^{2}(\mathbb{Q} \cap [0,1])$, while player II selects binary numbers $e_n \in \{0,1\}$:

$$
\begin{array}{llllllll}
\mathrm{I}: & h_0 & & h_1 & & h_2 & & \ldots \\
\mathrm{II}: & & e_0 & & e_1 & & e_2 & & \ldots
\end{array}
$$

At this point a short reminder of the notation seems appropriate: We denote pairs of rational numbers as functions $\{0,1\} = 2 \to \mathbb{Q}$ and we accordingly can write $h_n = \langle h_n(0), h_n(1)\rangle$.

Now consider a play $z = \langle h_0, e_0, h_1, e_1, \ldots\rangle$ of $\mathcal{G}_{v,X}$. For each $n \in \omega$ we impose the following rules each player must follow:

(a) $\frac{1}{2}h_n(0) + \frac{1}{2}h_n(1) \geq v_n$.

(b) $h_n(e_n) \neq 0$.

where we recursively defined $v_0 := v$ and $v_{n+1} := h_n(e_n)$.

An interpretation of these rules can be found later in this section. The play z is a win for player I if and only if $\langle e_n : n \in \omega \rangle \in X$. Note that only the moves made by player II are taken into account when determining the winner.

Both players must follow rule (a) and (b). However, to avoid undefined behavior when discussing strategies, it can be helpful to consider plays where some rules are violated. For example, if player I has a strategy σ and some sequence $y \in {}^\omega 2$ is given, $\sigma * y$ might be a play where some rules are broken.

Definition 2.4. *We call a play z of $\mathcal{G}_{v,X}$ legal if both players follow the rules (a) and (b). If z is legal we also call $z|_n$ legal for all $n \in \omega$.*

Note that the values of v_n and h_n are based on the play z of $\mathcal{G}_{v,X}$. However, the current notation doesn't make this dependence clear, so we introduce the following more robust version:

Definition 2.5. *Consider a play $z = \langle h_0, e_0, h_1, e_1, \ldots \rangle$ of $\mathcal{G}_{v,X}$ and v_n defined as above. For $n \in \omega$ we denote*

$$v^{z|_{2n}} = v^{z|_{2n+1}} := v_n \quad \text{and} \quad h^{z|_{2n+1}} := h_n.$$

Remark 2.6. *Note that the game $\mathcal{G}_{v,X}$ doesn't have the form $\mathcal{G}_A$ introduced in Section 1.2 for some $A \subseteq {}^\omega\omega$. However, with some changes in notation it is equivalent to a Game $\mathcal{G}_A$.*

First note that even without the axiom of choice there is a bijection $\omega \to {}^2\omega$ and a bijection $\omega \to \mathbb{Q} \cap [0,1]$. Thus there exists a bijection $q : {}^2(Q \cap [0,1]) \to \omega$.
In Section 1.2 the players where only allowed to choose natural numbers, so in $\mathcal{G}_{v,X}$ we will replace h_n with $q(h_n)$.

Now consider the set $A \subseteq {}^\omega\omega$ consisting of all plays z of $\mathcal{G}_{v,X}$ where one of the following occurred:

(i) player II *played some $e_n \notin \{0,1\}$*

(ii) player II *broke a rule before player* I *broke a rule*

(iii) both players didn't break rules and z is a win for I

Then the game $\mathcal{G}_{v,X}$ is identical, up to this change of notation, to $\mathcal{G}_A$. This approach further legitimizes considering illegal games as discussed before Definition 2.4

Now we examine how $\mathcal{G}_{v,X}$ plays out. Assume the sequence $r := \langle h_0, e_0, h_1, \ldots, e_{n-1} \rangle$ has already been played. Player II can choose any $e_n \in \{0,1\}$ unless player I selects h_n with $h_n(e) = 0$ for some $e \in \{0,1\}$, compelling player II to play $e_n := 1 - e$ by rule (b). In that case rule (a) ensures

$$\frac{1}{2}h_n(0) + \frac{1}{2}h_n(1) = \frac{1}{2}h_n(e_n) \geq v_n,$$

leading to $v_{n+1} = h_n(e_n) \geq 2v_n$. This potentially is disadvantageous for player I because, according to rule (a), they can only select a h_{n+1} containing a zero as long as $v_{n+1} \leq \frac{1}{2}$. This scenario illustrates further that it benefits player I to keep both values of h_n as small as rule (a) allows.

To better understand the purpose of $\mathcal{G}_{v,X}$ we discuss an interpretation of this game: For $e \in \{0,1\}$ consider the interval

$$I_n^e := g\big(\{x \in {}^\omega 2 : x|_{n+1} = \langle e_0, e_1, \ldots, e_{n-1}, e\rangle\}\big).$$

Note that I_n^0 and I_n^1 correspond to the left and right side of $I_{n-1}^{e_{n-1}}$, respectively.
In a play $z = r^\frown\langle h_n, e_n, h_{n+1}, e_{n+1}, \ldots\rangle$, the sequence $x = \langle e_0, e_1, \ldots\rangle$ determines the winner. By Lemma 2.1 it holds $g(x) \in I_n^{e_n}$, allowing player II to place $g(x)$ in either I_n^0 or I_n^1 by choosing the corresponding $e_n \in \{0,1\}$.

Assume for a moment that $g(X)$ is measurable. Intuitively, $\mu(g(X)) = \frac{\mu(g(X))}{\mu(g({}^\omega 2))}$ assigns a value to the otherwise undefined ratio $\frac{|X|}{|{}^\omega 2|}$. For example, if $X = \{x \in {}^\omega 2 : x(0) = 0\}$, then $\mu(g(X)) = \mu([0, \frac{1}{2}]) = \frac{1}{2}$ suggests that half of all infinite $\{0,1\}$ sequences start with 0.

Under this interpretation, $\mu\big(g(X) \cap I_n^e\big)$ measures the ratio

$$\frac{|\{x \in X : x|_{n+1} = \langle e_0, e_1, \ldots, e_{n-1}, e\rangle\}|}{|{}^\omega 2|},$$

and $2^{n+1}\mu\big(g(X) \cap I_n^e\big)$ measures

$$\frac{|\{x \in X : x|_{n+1} = \langle e_0, e_1, \ldots, e_{n-1}, e\rangle\}|}{|\{x \in {}^\omega 2 : x|_{n+1} = \langle e_0, e_1, \ldots, e_{n-1}, e\rangle\}|}.$$

We interpret the moves h_n and e_n as follows: player I claims lower bounds

$$2^{n+1}\mu\big(g(X) \cap I_n^e\big) \geq h_n(e)$$

for each $e \in \{0,1\}$. By selecting $e_n \in \{0,1\}$, player II asserts that this bound does not hold for e_n. Rule (b) prevents challenges to zero bounds, while rule (a) ensures, if the lower bounds from player I are correct, that

$$v_n \leq \frac{1}{2}\sum_{e \in \{0,1\}} h_n(e) \leq 2^n \sum_{e \in \{0,1\}} \mu\big(g(X) \cap I_n^e\big) = 2^n \mu\big(g(X) \cap I_{n-1}^{e_{n-1}}\big).$$

For example assume $v \leq \frac{1}{4}$ and X is the set of all $\{0,1\}$ sequences starting with $\langle 0,1\rangle$. Player I can always win by selecting $h_0 = \langle \frac{1}{2}, 0\rangle$ and $h_1 = \langle 0, 1\rangle$. Rule (b) forces player II to choose $e_0 = 0$ and $e_1 = 1$, ensuring player I wins. The move h_0 can be interpreted as claiming that at least half the sequences from ${}^\omega 2$ starting with 0 are contained in X, and h_1 as claiming that all sequences from ${}^\omega 2$ starting with $\langle 0,1\rangle$ are contained in X.

We now want to give some intuition for the next two lemmas in the upcoming sections, where $g(X)$ is not necessarily measurable. If player I has a winning strategy, their lower bounds h_n must always be accurate. For $n = 0$ this means

$$v = v_0 \leq 2^0 \mu_*(g(X)),$$

since $\mu_*(g(X))$ represents the largest lower bound for the measure of $g(X)$.

Conversely, if player II has a winning strategy, player I cannot provide accurate lower bounds, not even for $\mu^*(g(X))$, so

$$v = v_0 \geq \mu^*(g(X)).$$

2.3 Lower bound

Fix some $X \subseteq {}^{\omega}2$ and $v \in (0,1]$.

Lemma 2.7. *If player* I *has a winning strategy* σ *for* $\mathcal{G}_{v,X}$, *then* $\mu_*(g(X)) \geq v$.

Proof. Assume that player I has winning strategy σ for $\mathcal{G}_{v,X}$. For $\alpha \in \omega \cup \{\omega\}$ we define

$$A_\alpha := \{x \in {}^{\alpha}2 : \sigma * x \text{ is legal}\},$$

which is the set of all legal sequences of length α of moves by player II, assuming player I uses strategy σ. For $n \in \omega$ we additionally define the set C_n of infinite $\{0,1\}$ sequences with initial segment from A_n:

$$C_n := \{x \in {}^{\omega}2 : x|_n \in A_n\} = \bigsqcup_{p \in A_n} \{x \in {}^{\omega}2 : x|_n = p\}.$$

By Lemma 2.1 we get that $g(C_n)$ is a union of $|A_n|$ intervals of length 2^{-n} which pairwise intersect in at most one point. Thus $g(C_n)$ is measurable with $\mu(g(C_n)) = |A_n|2^{-n}$.

It holds by definition that $C_n \supseteq C_{n+1}$ for all $n \in \omega$, and further

$$A_\omega = \{x \in {}^{\omega}2 : \forall n \in \omega : \sigma * x|_n \text{ is legal}\} = \bigcap_{n \in \omega} C_n.$$

Using Corollary 2.3 we conclude that $g(A_\omega)$ is measurable with

$$\mu(g(A_\omega)) = \lim_{n \to \infty} \mu(g(C_n)) = \lim_{n \to \infty} |A_n|2^{-n}.$$

Since σ is a winning strategy, $A_\omega \subseteq X$, and thus $g(A_\omega) \subseteq g(X)$. By Fact 1.4, we conclude

$$\lim_{n \to \infty} |A_n|2^{-n} = \mu(g(A_\omega)) \leq \mu_*(g(X)). \tag{6}$$

It is left to prove that $v \leq \lim_{n \to \infty} |A_n|2^{-n}$. To achieve this we investigate the following function:

$$f : \bigcup_{n \in \omega} {}^{n}2 \to [0,1], p \mapsto \begin{cases} v^{\sigma * p} & \text{if } p \in \bigcup_{n \in \omega} A_n, \\ 0 & \text{otherwise.} \end{cases}$$

Claim. *For all* $n \in \omega$ *it holds*

$$2^{-n} \sum_{p \in {}^{n}2} f(p) \geq v.$$

Proof. We prove this statement by induction on n. The claim holds for $n = 0$ as the unique sequence of length 0 is legal and f maps it to v. Thus

$$2^{-0} \sum_{p \in {}^{0}2} f(p) = v.$$

Now assume the claim holds for some $n \in \omega$. First consider any $p \in A_n$. For $e \in \{0,1\}$ with $p^\frown\langle e \rangle \in A_{n+1}$ it holds by definition

$$f(p^\frown\langle e \rangle) = v^{\sigma * (p^\frown\langle e \rangle)} = h^{\sigma * p}(e).$$

Conversely, for $e \in \{0,1\}$ with $p^\frown\langle e\rangle \notin A_n$ player II breakes rule (b) when extending $\sigma * p$ by e, and consequently

$$f(p^\frown\langle e\rangle) = 0 = h^{\sigma * p}(e).$$

By rule (a) we conclude that for all $p \in A_n$:

$$\frac{1}{2}f(p^\frown\langle 0\rangle) + \frac{1}{2}f(p^\frown\langle 1\rangle) = \frac{1}{2}h^{\sigma*p}(0) + \frac{1}{2}h^{\sigma*p}(1) \geq v^{\sigma*p} = f(p).$$

On the other hand, for all $p \in {}^n2 \setminus A_n$ we have $p^\frown\langle 0\rangle, p^\frown\langle 1\rangle \notin A_{n+1}$, and thus

$$\frac{1}{2}f(p^\frown\langle 0\rangle) + \frac{1}{2}f(p^\frown\langle 1\rangle) = 0 \geq 0 = f(p).$$

So by the induction hypothesis and the fact that

$${}^{n+1}2 = \{p^\frown\langle 0\rangle : p \in {}^n2\} \sqcup \{p^\frown\langle 1\rangle : p \in {}^n2\},$$

we conclude

$$\begin{aligned} 2^{-(n+1)} \sum_{p \in {}^{n+1}2} f(p) &= 2^{-(n+1)} \sum_{p \in {}^n2} \big(f(p^\frown\langle 0\rangle) + f(p^\frown\langle 1\rangle)\big) \\ &= 2^{-n} \sum_{p \in {}^n2} \Big(\frac{1}{2}f(p^\frown\langle 0\rangle) + \frac{1}{2}f(p^\frown\langle 1\rangle)\Big) \\ &\geq 2^{-n} \sum_{p \in {}^n2} f(p) \geq v. \end{aligned}$$

$\dashv_{\text{Claim}}$

Fix some arbitrary $n \in \omega$. Using that $f(p) = 0$ if $p \notin A_n$ and $f(p) \leq 1$ for all $p \in {}^n2$ we deduce from the claim

$$v \leq 2^{-n} \sum_{p \in {}^n2} f(p) = 2^{-n} \sum_{p \in A_n} f(p) \leq 2^{-n} \sum_{p \in A_n} 1 = |A_n| 2^{-n}.$$

Taking the limit as $n \to \infty$ we conclude by equation (6)

$$v \leq \lim_{n\to\infty} |A_n| 2^{-n} \leq \mu_*(g(X)).$$

$\square$

2.4 Upper bound

Fix some $X \subseteq {}^{\omega}2$ and $v \in (0,1]$.

Lemma 2.8. *If player* II *has a winning strategy* τ *for* $\mathcal{G}_{v,X}$*, then* $\mu^*(g(X)) \leq v$.

Proof. Assume that player II has winning strategy τ and fix an arbitrary $\delta > 0$.

We will recursively construct sets $A_n \subseteq {}^{n}2$ consisting of sequences of moves by player II utilizing strategy τ and functions

$$\psi_n : A_n \to \{z|_{2n} : z \text{ is a play of } \mathcal{G}_{v,X} \text{ consistent with } \tau\},$$

which complete these sequences with the moves by player I. For $n \in \omega$ and $p \in {}^{n}2$, the function ψ_n should select moves for player I that are nearly optimal for them, meaning $\psi_n(p)$ almost minimizes $v^{\psi_n(p)}$ among all legal sequences which are consistent with τ where player II makes the moves p. We will exclude p from A_n and not even define $\psi_n(p)$ if we would have $v^{\psi_n(p)} = 1$ or if there is no legal $\psi_n(p)$ consistent with τ.

Define $A_0 := {}^{0}2$ and let ψ_0 be the identity on A_0.
Now assume A_n and ψ_n are already constructed for some $n \in \omega$. Fix $p \in A_n$ and $e \in \{0,1\}$. We extend $\psi_n(p)$ by a move a^p_e of player I and the move e of player II such that $\psi_n(p)^\frown\langle a^p_e, e\rangle$ fulfills the criteria for $\psi_{n+1}(p^\frown\langle e\rangle)$ as discussed above. So we define

$$S^p_e := \{h \in {}^{2}(\mathbb{Q} \cap [0,1]) : \psi_n(p)^\frown\langle h\rangle \text{ is legal and } \tau(\psi_n(p)^\frown\langle h\rangle) = e\},$$

which is the set of all legal moves by player I such that player II chooses e next when playing with τ. The next move a^p_e by player I should be chosen from S^p_e such that $v^{\psi_n(p)^\frown\langle a^p_e, e\rangle}$ is almost minimized. This gives rise to the function

$$u^p : \{0,1\} \to [0,1], \quad m \mapsto \inf\big(\{h(m) : h \in S^p_m\} \cup \{1\}\big).$$

Note that $u^p(e) = 1$ means that if player II uses strategy τ, they either never continue a legal sequence $\psi_n(p)^\frown\langle h\rangle$ with e, or only do so if $h(e) = 1$. Therefore we set

$$A_{n+1} := \{p^\frown\langle m\rangle : m \in \{0,1\} \text{ and } u^p(m) \neq 1\}.$$

If $p^\frown\langle e\rangle \in A_{n+1}$, we choose some $a^p_e \in S^p_e$ with $a^p_e(e) \leq u^p(e) + 2^{-(n+1)}\delta$ as the next move of player I. To address the absence of the axiom of choice, note that there exists a bijection $q : {}^{2}(\mathbb{Q} \cap [0,1]) \to \omega$, allowing us to choose the a^p_e with minimal $q(a^p_e)$. Finally, we define

$$\psi_{n+1}(p^\frown\langle e\rangle) = \psi_n(p)^\frown\langle a^p_e, e\rangle,$$

which meets the criteria for ψ_{n+1} discussed before.

Consider the set

$$A_\omega := \{x \in {}^{\omega}2 : \forall n \in \omega : x|_n \in A_n\}.$$

Note that for each $x \in A_\omega$ we find a play z of $\mathcal{G}_{v,X}$ such that $z|_{2n} = \psi_n(x)$ for all $n \in \omega$. This z is clearly consistent with the winning strategy τ, and consequently $x \notin X$. So, we conclude $A_\omega \subseteq {}^{\omega}2 \setminus X$, or equivalently $X \subseteq {}^{\omega}2 \setminus A_\omega$.

For $\alpha \in \omega \cup \{\omega\}$ consider the set D_α of infinite $\{0,1\}$ sequences with initial segment **not** in A_α:

$$D_\alpha := \{x \in {}^\omega 2 : x|_\alpha \notin A_\alpha\} = \bigsqcup_{p \in {}^\alpha 2 \setminus A_\alpha} \{x \in {}^\omega 2 : x|_\alpha = p\}.$$

For $n \in \omega$ we get by Lemma 2.1 that $g(D_n)$ is a union of $2^n - |A_n|$ intervals of length 2^{-n} which pairwise intersect in at most one point. Thus $g(D_n)$ is measurable with

$$\mu(g(D_n)) = (2^n - |A_n|) \cdot 2^{-n} = 1 - |A_n| 2^{-n}.$$

Note that by construction $D_0 \subseteq D_1 \subseteq D_2 \subseteq \dots$ and further

$$X \subseteq {}^\omega 2 \setminus A_\omega = D_\omega = \{x \in {}^\omega 2 : \exists n \in \omega : x \notin A_n\} = \bigcup_{n \in \omega} D_n.$$

Therefore $g(D_\omega) = g\big(\bigcup_{n\in\omega} D_n\big) = \bigcup_{n\in\omega} g(D_n)$ and we get by Fact 1.5 that $g(D_\omega)$ is measurable with

$$\mu(g(D_\omega)) = \lim_{n\to\infty} \mu(g(D_n)) = \lim_{n\to\infty} \big(1 - |A_n|2^{-n}\big).$$

Using Fact 1.4 we deduce from $g(X) \subseteq g(D_\omega)$ that

$$\mu^*(g(X)) \le g(D_\omega) = \lim_{n\to\infty} \big(1 - |A_n|2^{-n}\big). \tag{7}$$

To get an estimation of the this limit we investigate the following functions for $n \in \omega$:

$$f_n : {}^n 2 \to [0,1], p \mapsto \begin{cases} v^{\psi_n(p)} & \text{if } p \in A_n, \\ 1 & \text{otherwise.} \end{cases}$$

Claim 1. *For all $n \in \omega$ and $p \in A_n$ it holds*

$$\frac{1}{2} u^p(0) + \frac{1}{2} u^p(1) \le v^{\psi_n(p)}.$$

Proof. Assume by contradiction that there exists $\varepsilon > 0$ such that

$$\frac{1}{2} u^p(0) + \frac{1}{2} u^p(1) - \varepsilon = v^{\psi_n(p))}.$$

Then there exists some $h : \{0,1\} \to (\mathbb{Q} \cap [0,1])$ such that for all $e \in \{0,1\}$

$$h(e) \in \begin{cases} \{0\} & \text{if } u^p(e) = 0, \\ (u^p(e), u^p(e) - \varepsilon) & \text{otherwise.} \end{cases}$$

Note that $\psi_n(p)^\frown\langle h\rangle$ is legal since $\psi_n(p)$ is legal and

$$\frac{1}{2} \sum_{e \in \{0,1\}} h(e) \ge \frac{1}{2} \sum_{e \in \{0,1\}} \big(u^p(e) - \varepsilon\big) = v^{\psi_n(p)}.$$

Now consider $e := \tau(\psi_n(p)^\frown\langle h\rangle)$. Clearly by the definition of u^p it holds $u^p(e) \le h(e)$ and therefore $u^p(e) = h(e) = 0$. Consequently $\psi_n(p)^\frown\langle h, e\rangle$ is consistent with τ but violates rule (b), contradicting the fact that τ is a winning strategy. $\dashv_{\text{Claim}}$

Claim 2. *For all $n \in \omega$ it holds*

$$2^{-n} \sum_{p \in {}^{n}2} f_n(p) \leq v + \frac{2^n - 1}{2^n}\delta.$$

Proof. We prove this statement by induction on n. The claim holds for $n = 0$ as f_0 map the sequence of length 0 to v and therefore

$$2^{-0} \sum_{p \in {}^{0}2} f(p) = v = v + \frac{2^0 - 1}{2^0}\delta.$$

Now assume the statement holds for some $n \in \omega$. First consider an arbitrary $p \in A_n$.
For all $e \in \{0, 1\}$ with $p^\frown\langle e\rangle \in A_{n+1}$ it holds by definition

$$f_{n+1}(p^\frown\langle e\rangle) = v^{\psi_n(p^\frown\langle e\rangle)} = a_e^p(e) \leq u^p(e) + 2^{-(n+1)}\delta.$$

On the other hand, for all $e \in \{0, 1\}$ with $p^\frown\langle e\rangle \notin A_{n+1}$ it holds

$$f_{n+1}(p^\frown\langle e\rangle) = 1 = u^p(e) \leq u^p(e) + 2^{-(n+1)}\delta.$$

So we conclude by Claim 1 that for all $p \in A_n$ we have

$$\frac{1}{2} \sum_{e \in \{0,1\}} f_{n+1}(p^\frown\langle e\rangle) \leq \frac{1}{2} \sum_{e \in \{0,1\}} \big(u^p(e) + 2^{-(n+1)}\delta\big) \leq v^{\psi_n(p)} + 2^{-(n+1)}\delta = f_n(p) + \frac{\delta}{2^{n+1}}.$$

Conversely, for $p \notin A_n$ we get $p^\frown\langle 0\rangle, p^\frown\langle 1\rangle \notin A_{n+1}$ and thus

$$\frac{1}{2} \sum_{e \in \{0,1\}} f_{n+1}(p^\frown\langle e\rangle) = \frac{1}{2} + \frac{1}{2} \leq 1 + \frac{\delta}{2^{n+1}} = f_n(p) + \frac{\delta}{2^{n+1}}.$$

So by the induction hypothesis and the fact

$${}^{n+1}2 = \{p^\frown\langle 0\rangle : p \in {}^{n}2\} \sqcup \{p^\frown\langle 1\rangle : p \in {}^{n}2\},$$

we conclude

$$\begin{aligned} 2^{-(n+1)} \sum_{p \in {}^{n+1}2} f_{n+1}(p) &= 2^{-n} \sum_{p \in {}^{n}2} \frac{1}{2} \sum_{e \in \{0,1\}} f_{n+1}(p^\frown\langle e\rangle) \\ &\leq 2^{-n} \sum_{p \in {}^{n}2} \big(f_n(p) + \frac{\delta}{2^{n+1}}\big) \\ &\leq v + \frac{2^n - 1}{2^n}\delta + \frac{\delta}{2^{n+1}} \\ &= v + \frac{2 \cdot (2^n - 1) + 1}{2^{n+1}}\delta \\ &= v + \frac{2^{n+1} - 1}{2^{n+1}}\delta. \end{aligned}$$

$\dashv_{\text{Claim}}$

Fix some arbitrary $n \in \omega$. Using that $f(p) = 1$ if $p \notin A_n$ and $f(p) \geq 0$ for $p \in {}^n2$ we deduce from Claim 2

$$1 - 2^{-n}|A_n| = 2^{-n}(2^n - |A_n|) = 2^{-n} \sum_{p \in {}^n2 \setminus A_n} 1 \leq 2^{-n} \sum_{p \in {}^n2} f_n(p) \leq v + \frac{2^n - 1}{2^n}\delta \leq v + \delta.$$

Taking the limit as $n \to \infty$ we conclude by equation (7)

$$\mu^*(g(X)) \leq \lim_{n \to \infty} \left(1 - |A_n|2^{-n}\right) \leq v + \delta.$$

Since this holds for all $\delta > 0$ the lemma follows. □

2.5 Proving measurability

Having established the key Lemmas 2.7 and 2.8, we can now easily demonstrate, assuming AD holds, that all subsets of $[0, 1]$ are measurable.

Theorem 2.9. *The axiom of determinacy implies that all sets $Y \subseteq [0, 1]$ are measurable.*

Proof. We proof this by contraposition. Assume there exists a set $Y \subseteq [0, 1]$ which is not measurable. Then clearly

$$0 \leq \mu_*(Y) < \mu^*(Y) \leq 1,$$

and thus there exists $v \in (0, 1]$ with

$$\mu_*(Y) < v < \mu^*(Y). \tag{8}$$

.

Because g is surjective there exists $X \subseteq {}^{\omega}2$ with $g(X) = Y$. If player I has a winning strategy for $\mathcal{G}_{v,X}$ we get by Lemma 2.7 that $v \leq \mu_*(Y)$ which contradicts inequality (8). Conversely, if player II has a winning strategy for $\mathcal{G}_{v,X}$ we get by Lemma 2.8 that $\mu^*(Y) \leq v$ which also contradicts inequality (8).

Thus neither player I nor player II has a winning strategy for $\mathcal{G}_{v,X}$. Consequently the game $\mathcal{G}_{v,X}$ is not determined, so AD doesn't hold. □

Finally we can generalize this result to all subsets of $\mathbb{R}$.

Corollary 2.10. *The axiom of determinacy implies that all sets $Y \subseteq \mathbb{R}$ are measurable.*

Proof. Let $Y \subseteq \mathbb{R}$ be arbitrary. For each $n \in \mathbb{Z}$ we define $Y_n := Y \cap [n, n+1]$. For all $n \in \mathbb{Z}$ we conclude from Theorem 2.9 that $Y_n - n := \{y - n : y \in Y_n\} \subseteq [0, 1]$ is measurable and thus, by the translation invariance of the Lebesgue measure, Y_n is also measurable. Therefore

$$Y = \{y \in Y : \exists n \in \mathbb{Z}\, (n \leq y \leq n + 1)\} = \bigcup_{n \in \mathbb{Z}} Y_n$$

is measurable by Fact 1.5. □

3 Discussion

In this thesis, we have presented a proof that the axiom of determinacy (AD) implies that all sets of real numbers are Lebesgue measurable, following the approach from Martin's work "A simple proof that determinacy implies Lebesgue measurability." [4]. The game $\mathcal{G}_{v,X}$ was defined, where two players attempt to estimate the measure of a set by recursively splitting intervals in half.

Through two extensive lemmas, it was established that if one player has a winning strategy, this corresponds to an estimation of the inner or outer measure of the set, respectively. Together these two lemmas yield that under AD, no subsets of $[0,1]$ can exist with an inner measure that is strictly smaller than their outer measure, implying that all subsets of $[0,1]$ are indeed measurable. We then extended this result to subsets of $\mathbb{R}$ by using the translation invariance of the Lebesgue measure.

This method is an accessible approach to proving the statement, in contrast to the more commonly employed techniques involving the definition and analysis of analytic sets. However, Martin's method does not immediately provide the more general result that every subset of $\mathbb{R}^m$ is Lebesgue measurable.

To address this limitation and extend the result to higher dimensions, we can adapt the game and the function g as follows: We define

$$\tilde{g} : {}^{\omega}({}^{m}2) \to [0,1]^m, x \mapsto \langle \sum_{n=0}^{\infty} x(n)(k) : k \in m \rangle$$

and modify the game $\mathcal{G}_{v,X}$ accordingly. Fix some $X \subseteq {}^{\omega}({}^{m}2)$ and $v \in (0,1]$. Player I plays functions $h_n : {}^{m}2 \to \mathbb{Q} \cap [0,1]$, while player II plays sequences $e_n \in {}^{m}2$.

$$\begin{array}{llllllll} \text{I}: & h_0 & & h_1 & & h_2 & & \dots \\ \text{II}: & & e_0 & & e_1 & & e_2 & & \dots \end{array}$$

For each $n \in \omega$, the rules are:

(a) $2^{-m} \sum_{p \in {}^{m}2} h_n(p) \geq v_n$.

(b) $h_n(e_n) \neq 0$.

where $v_0 := v$ and $v_{n+1} := h_n(e_n)$ is defined by recursion. A play is a win for I if and only if $\langle e_n : n \in \omega \rangle \in X$.

With these adjustments, Lemma 2.7 and Lemma 2.8 remain valid as shown in Appendix A and B. Consequently, through analogous arguments as presented in Section 2.5, we can conclude that AD implies that all subsets of $\mathbb{R}^m$ are measurable.

The fact that the axiom of determinacy implies all sets of reals are measurable demonstrates that an alternative foundation of set theory can improve regularity in certain aspects of measure theory, avoiding some pathological constructions that arise from the axiom of choice. There are more fascinating regularity properties, for example the property of Baire, which are also implied by AD. However, these topics extend beyond the scope of this thesis and are not covered here.

Appendices

A Lower bound for the multidimensional case

In this appendix we want to prove a generalisation of Lemma 2.7 to sets of $\mathbb{R}^m$.

Fix some $X \subseteq {}^{\omega}({}^{m}2)$ and $v \in (0,1]$ and consider the Game $\mathcal{G}_{v,X}$ as defined in section 3.

Lemma A.1. *If player* I *has a winning strategy* σ *for* $\mathcal{G}_{v,X}$ *then* $\mu_*(\tilde{g}(X)) \geq v$.

Proof. Assume that player I has winning strategy σ. For $\alpha \in \omega \cup \{\omega\}$ we define

$$A_\alpha := \{x \in {}^{\alpha}({}^{m}2) : \sigma * x \text{ is legal}\}.$$

For $n \in \omega$ we additionally define

$$C_n := \{x \in {}^{\omega}({}^{m}2) : x|_n \in A_n\} = \bigsqcup_{p \in A_n} \{x \in {}^{\omega}({}^{m}2) : x|_n = p\}.$$

Analogically to Lemma 2.1 we get that $\tilde{g}(C_n)$ is a union of $|A_n|$ multidimensional intervals of measure $(2^{-n})^m$ which pairwise intersect in some set of measure 0. Thus $\tilde{g}(C_n)$ is measurable with

$$\mu(\tilde{g}(C_n)) = |A_n| 2^{-mn}.$$

It holds by definition that $C_n \supseteq C_{n+1}$ for all $n \in \omega$, and further

$$A_\omega = \{x \in {}^{\omega}({}^{m}2) : \forall n \in \omega : \sigma * x|_n \text{ is legal}\} = \bigcap_{n \in \omega} C_n.$$

By a modified version of Corollary 2.3 we conclude that $\tilde{g}(A_\omega) = \bigcap_{n\in\omega} \tilde{g}(X_n)$ is measurable with

$$\mu(\tilde{g}(A_\omega)) = \lim_{n\to\infty} \mu(\tilde{g}(C_n)) = \lim_{n\to\infty} |A_n| 2^{-mn}.$$

Since σ is a winning strategy it holds $A_\omega \subseteq X$ and thus $\tilde{g}(A_\omega) \subseteq \tilde{g}(X)$. By Fact 1.4 we conclude

$$\lim_{n\to\infty} |A_n| 2^{-mn} = \mu(\tilde{g}(A_\omega)) \leq \mu_*(\tilde{g}(X)). \tag{9}$$

It is left to prove that $v \leq \lim_{n\to\infty} |A_n| 2^{-mn}$. To achieve this we investigate the following function:

$$f : \bigcup_{n\in\omega} {}^{n}({}^{m}2) \to [0,1], p \mapsto \begin{cases} v^{\sigma * p} & \text{if } p \in \bigcup_{n\in\omega} A_n, \\ 0 & \text{otherwise.} \end{cases}$$

Claim. *For all $n \in \omega$ it holds*

$$2^{-mn} \sum_{p \in {}^n({}^m 2)} f(p) \geq v.$$

Proof. We prove this statement by induction on n. The claim holds for $n = 0$ as the unique sequence of length 0 is legal and f maps it to v. Thus

$$2^{-0} \sum_{p \in {}^0({}^m 2)} f(p) = v.$$

Now assume the claim holds for some $n \in \omega$. First consider any $p \in A_n$. For $q \in {}^m 2$ with $p^\frown\langle q \rangle \in A_{n+1}$ it holds by definition

$$f(p^\frown\langle q \rangle) = v^{\sigma * (p^\frown\langle q \rangle)} = h^{\sigma * p}(q).$$

Conversely, for $q \in {}^m 2$ with $p^\frown\langle q \rangle \notin A_n$ player II breakes rule (b) when extending $\sigma * p$ with q, and consequently

$$f(p^\frown\langle q \rangle) = 0 = h^{\sigma * p}(q).$$

By rule (a) we conclude that for all $p \in A_n$

$$2^{-m} \sum_{q \in {}^m 2} f(p^\frown\langle q \rangle) = 2^{-m} \sum_{q \in {}^m 2} h^{\sigma * p}(q) \geq v^{\sigma * p} = f(p).$$

On the other hand, for all $p \in {}^n({}^m 2) \setminus A_n$ we have $p^\frown\langle q \rangle \notin A_{n+1}$ for all $q \in {}^m 2$, and thus

$$2^{-m} \sum_{q \in {}^m 2} f(p^\frown\langle q \rangle) = 0 = f(p).$$

So by the induction hypothesis and the fact

$${}^{n+1}({}^m 2) = \bigsqcup_{q \in {}^m 2} \{p^\frown\langle q \rangle : p \in {}^n({}^m 2)\},$$

we conclude

$$\begin{aligned} 2^{-m(n+1)} \sum_{p \in {}^{n+1}({}^m 2)} f(p) &= 2^{-mn} \sum_{p \in {}^n({}^m 2)} 2^{-m} \sum_{q \in {}^m 2} f(p^\frown\langle q \rangle) \\ &\geq 2^{-mn} \sum_{p \in {}^n({}^m 2)} f(p) \geq v. \end{aligned}$$

$\dashv_{\text{Claim}}$

Fix some arbitrary $n \in \omega$. Using that $f(p) = 0$ if $p \notin A_n$ and $f(p) \leq 1$ for all $p \in {}^n({}^m 2)$ we deduce from the claim that

$$v \leq 2^{-mn} \sum_{p \in {}^n({}^m 2)} f(p) = 2^{-mn} \sum_{p \in A_n} f(p) \leq 2^{-mn} \sum_{p \in A_n} 1 = |A_n| 2^{-mn}.$$

Taking the limit as $n \to \infty$ we conclude by equation (9)

$$v \leq \lim_{n \to \infty} |A_n| 2^{-mn} \leq \mu_*(g(X)).$$

$\square$

B Upper bound for the multidimensional case

In this appendix we want to prove a generalisation of Lemma 2.8 to sets of $\mathbb{R}^m$.

Fix some $X \subseteq {}^{\omega}({}^{m}2)$ and $v \in (0,1]$ and consider the Game $\mathcal{G}_{v,X}$ as defined in section 3.

Lemma B.1. *If player* II *has a winning strategy* τ *for* $\mathcal{G}_{v,X}$ *then* $\mu^*(\tilde{g}(X)) \leq v$.

Proof. Assume that player II has winning strategy τ and fix an arbitrary $\delta > 0$.

We will recursively construct sets $A_n \subseteq {}^{n}({}^{m}2)$ consisting of sequences of moves by player II utilizing strategy τ and functions

$$\psi_n : A_n \to \{z|_{2n} : z \text{ is a play of } \mathcal{G}_{v,X} \text{ consistent with } \tau\},$$

which complete these sequences with the moves by player I. For $n \in \omega$ and $p \in {}^{n}({}^{m}2)$, the function ψ_n should select moves for player I that are nearly optimal for them, meaning $\psi_n(p)$ almost minimizes $v^{\psi_n(p)}$ among all legal sequences which are consistent with τ where player II makes the moves p. We will exclude p from A_n and not even define $\psi_n(p)$ if we would have $v^{\psi_n(p)} = 1$ or if there is no legal $\psi_n(p)$ consistent with τ.

Define $A_0 := {}^{0}({}^{m}2)$ and let ψ_0 be the identity on A_0.
Now assume A_n and ψ_n are already constructed for some $n \in \omega$. Fix $p \in A_n$ and $q \in {}^{m}2$. We extend $\psi_n(p)$ by a move a^p_q of player I and the move q of player II such that $\psi_n(p)^\frown\langle a^p_q, q\rangle$ fulfills the criteria for $\psi_{n+1}(p^\frown\langle q\rangle)$ as discussed above. So we define

$$S^p_q := \{h \in {}^{({}^{m}2)}(\mathbb{Q} \cap [0,1]) : \psi_n(p)^\frown\langle h\rangle \text{ is legal and } \tau(\psi_n(p)^\frown\langle h\rangle) = q\},$$

which is the set of all legal moves by player I such that player II chooses q next when playing with τ. The next move a^p_q by player I should be chosen from S^p_q such that $v^{\psi_n(p)^\frown\langle a^p_q, q\rangle}$ is almost minimized. This gives rise to the function

$$u^p : {}^{m}2 \to [0,1], \quad r \mapsto \inf\big(\{h(r) : h \in S^p_r\} \cup \{1\}\big).$$

Note that $u^p(q) = 1$ means that if player II uses strategy τ, they either never continue a legal sequence $\psi_n(p)^\frown\langle h\rangle$ with q or only if $h(q) = 1$. So we set

$$A_{n+1} := \{p^\frown\langle r\rangle : r \in {}^{m}2 \text{ and } u^p(r) \neq 1\}.$$

If $p^\frown\langle q\rangle \in A_{n+1}$, we choose some $a^p_q \in S^p_q$ with $a^p_q(q) \leq u^p(q) + 2^{-m(n+1)}\delta$ as the next move of player I. This can be done without the axiom of choice as ${}^{({}^{m}2)}\mathbb{Q}$ is countable. Finally, we define

$$\psi_{n+1}(p^\frown\langle q\rangle) = \psi_n(p)^\frown\langle a^p_q, q\rangle,$$

which meets the criteria for ψ_{n+1} discussed before.

Consider the set

$$A_\omega := \{x \in {}^{\omega}({}^{m}2) : \forall n \in \omega : x|_n \in A_n\}.$$

Note that for each $x \in A_\omega$ we find a play z of $\mathcal{G}_{v,X}$ such that $z|_{2n} = \psi_n(x)$ for all $n \in \omega$. This z is clearly consistent with the winning strategy τ and consequently $x \notin X$. So, we conclude $A_\omega \subseteq {}^{\omega}({}^{m}2) \setminus X$ or equivalently $X \subseteq {}^{\omega}({}^{m}2) \setminus A_\omega$.

For $\alpha \in \omega \cup \{\omega\}$ consider the set D_α of infinite ${}^m2$ sequences with initial segment **not** in A_α:

$$D_\alpha := \{x \in {}^\omega({}^m2) : x|_\alpha \notin A_\alpha\} = \bigsqcup_{p \in {}^\alpha({}^m2) \setminus A_\alpha} \{x \in {}^\omega({}^m2) : x|_\alpha = p\}.$$

Analogically to Lemma 2.1 we get that $\tilde{g}(D_n)$ is a union of $2^{mn} - |A_n|$ multidimensional intervals of measure $(2^{-n})^m$ which pairwise intersect in some set of measure 0. So $\tilde{g}(D_n)$ is measurable with

$$\mu(\tilde{g}(D_n)) = \big(2^{mn} - |A_n|\big)2^{-mn} = 1 - |A_n|2^{-mn}.$$

Note that by construction $D_0 \subseteq D_1 \subseteq D_2 \subseteq \ldots$ and further

$$X \subseteq {}^\omega({}^m2) \setminus A_\omega = D_\omega = \{x \in {}^\omega({}^m2) : \exists n \in \omega : x \notin A_n\} = \bigcup_{n \in \omega} D_n.$$

Therefore $\tilde{g}(D_\omega) = \tilde{g}\big(\bigcup_{n\in\omega} D_n\big) = \bigcup_{n\in\omega} \tilde{g}(D_n)$ and we get by a modified version of Fact 1.5 that $\tilde{g}(D_\omega)$ is measurable with

$$\mu(\tilde{g}(D_\omega)) = \lim_{n\to\infty} \mu(\tilde{g}(D_n)) = \lim_{n\to\infty} \big(1 - |A_n|2^{-mn}\big).$$

Using a modified version of Fact 1.4 we deduce from $\tilde{g}(X) \subseteq \tilde{g}(D_\omega)$ that

$$\mu^*(\tilde{g}(X)) \leq \tilde{g}(D_\omega) = \lim_{n\to\infty} \big(1 - |A_n|2^{-mn}\big). \tag{10}$$

To get an estimation of the this limit we investigate the following functions for $n \in \omega$:

$$f_n : {}^n({}^m2) \to [0,1], p \mapsto \begin{cases} v^{\psi_n(p)} & \text{if } p \in A_n, \\ 1 & \text{otherwise.} \end{cases}$$

Claim 1. *For all $n \in \omega$ and $p \in A_n$ it holds*

$$2^{-m} \sum_{q \in {}^m2} u^p(q) \leq v^{\psi_n(p)}.$$

Proof. Assume by contradiction that there exists $\varepsilon > 0$ such that

$$2^{-m} \sum_{q \in {}^m2} u^p(q) - \varepsilon = v^{\psi_n(p))}.$$

Then there exists some $h : {}^m2 \to (\mathbb{Q} \cap [0,1])$ such that for all $q \in {}^m2$

$$h(q) \in \begin{cases} \{0\} & \text{if } u^p(q) = 0, \\ (u^p(q), u^p(q) - \varepsilon) & \text{otherwise.} \end{cases}$$

Note that $\psi_n(p)^\frown\langle h\rangle$ is legal since $\psi_n(p)$ is legal and

$$2^{-m} \sum_{q \in {}^m2} h(q) \geq 2^{-m} \sum_{q \in {}^m2} \big(u^p(q) - \varepsilon\big) = v^{\psi_n(p)}.$$

Now consider $q := \tau(\psi_n(p)^\frown\langle h\rangle)$. Clearly by the definition of u^p it holds $u^p(q) \leq h(q)$ and therefore $u^p(q) = h(q) = 0$. Consequently, $\psi_n(p)^\frown\langle h, q\rangle$ is consistent with τ but violates rule (b), contradicting the fact that τ is a winning strategy. $\dashv_{\text{Claim}}$

Claim 2. *For all $n \in \omega$ it holds*

$$2^{-mn} \sum_{p \in {}^{n}({}^{m}2)} f_n(p) \leq v + \frac{2^{mn}-1}{2^{mn}}\delta.$$

Proof. We prove this statement by induction on n. The claim holds for $n = 0$ as f_0 map the sequence of length 0 to v and therefore

$$2^{-m\cdot 0} \sum_{p \in {}^{0}({}^{m}2)} f(p) = v = v + \frac{2^{m\cdot 0}-1}{2^{m\cdot 0}}\delta.$$

Now assume the statement holds for some $n \in \omega$. First consider an arbitrary $p \in A_n$.
For all $q \in {}^{m}2$ with $p^\frown\langle q\rangle \in A_{n+1}$ it holds by definition

$$f_{n+1}(p^\frown\langle q\rangle) = v^{\psi_n(p^\frown\langle q\rangle)} = a^p_q(q) \leq u^p(q) + 2^{-m(n+1)}\delta.$$

On the other hand, for all $q \in {}^{m}2$ with $p^\frown\langle q\rangle \notin A_{n+1}$ it holds

$$f_{n+1}(p^\frown\langle q\rangle) = 1 = u^p(q) \leq u^p(q) + 2^{-m(n+1)}\delta.$$

So we conclude by Claim 1 that for all $p \in A_n$ we have

$$2^{-m} \sum_{q \in {}^{m}2} f_{n+1}(p^\frown\langle q\rangle) \leq 2^{-m} \sum_{q \in {}^{m}2} \big(u^p(q) + 2^{-m(n+1)}\delta\big) \leq v^{\psi_n(p)} + \frac{\delta}{2^{m(n+1)}} = f_n(p) + \frac{\delta}{2^{m(n+1)}}.$$

Conversely, for $p \notin A_n$ and $q \in {}^{m}2$, we get $p^\frown\langle q\rangle \notin A_{n+1}$, and thus

$$2^{-m} \sum_{q \in {}^{m}2} f_{n+1}(p^\frown\langle q\rangle) = 1 \leq 1 + \frac{\delta}{2^{m(n+1)}} = f_n(p) + \frac{\delta}{2^{m(n+1)}}.$$

So by the induction hypothesis and the fact

$${}^{n+1}({}^{m}2) = \bigsqcup_{q \in {}^{m}2} \{p^\frown\langle q\rangle : p \in {}^{n}({}^{m}2)\},$$

we conclude

$$\begin{aligned}
2^{-m(n+1)} \sum_{p \in {}^{n+1}({}^{m}2)} f_{n+1}(p) &= 2^{-mn} \sum_{p \in {}^{n}({}^{m}2)} 2^{-m} \sum_{q \in {}^{m}2} f_{n+1}(p^\frown\langle q\rangle) \\
&\leq 2^{-n} \sum_{p \in {}^{n}({}^{m}2)} \big(f_n(p) + \frac{\delta}{2^{m(n+1)}}\big) \\
&\leq v + \frac{2^{mn}-1}{2^{mn}}\delta + \frac{\delta}{2^{m(n+1)}} \\
&= v + \frac{2^m \cdot (2^{mn}-1)+1}{2^{m(n+1)}}\delta \\
&\leq v + \frac{2^{m(n+1)}-1}{2^{m(n+1)}}\delta.
\end{aligned}$$

$\dashv_{\text{Claim}}$

Fix some arbitrary $n \in \omega$. Using that $f(p) = 1$ if $p \notin A_n$ and $f(p) \geq 0$ for $p \in {}^n({}^m2)$ we deduce from Claim 2

$$1 - 2^{-mn}|A_n| = 2^{-mn}(2^{mn} - |A_n|) = 2^{-mn} \sum_{p \in {}^n({}^m2) \setminus A_n} 1 \leq 2^{-mn} \sum_{p \in {}^n({}^m2)} f_n(p) \leq v + \frac{2^{mn} - 1}{2^{mn}} \delta \leq v + \delta.$$

Taking the limit as $n \to \infty$ we conclude by equation (10)

$$\mu^*(\tilde{g}(X)) \leq \lim_{n \to \infty} \left(1 - |A_n| 2^{-mn}\right) \leq v + \delta.$$

Since this holds for all $\delta > 0$ the Lemma follows. □

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