Question 1. (12 points)
Multiple choice questions: only the answer will be evaluated, and exactly one of the options is correct.

(a) (3 points) Let $f_j, f : [0,1] \to [0,\infty)$ be measurable functions (with respect to $\mathcal{L}^1$) for $j = 1, 2, \ldots$ and suppose that $f_j \to f$ almost everywhere. Which of the following statements is true?

(A) There exists a measurable subset $A \subset [0,1]$ with $\mathcal{L}^1([0,1] \setminus A) = 0$ such that $f_j \to f$ uniformly in A.
(B) If $\|f_j\|_{L^1([0,1])} = 1$ for each j, then $f \in L^1([0,1])$ with $\|f\|_{L^1([0,1])} = 1$.
(C) If $0 \leq f_1 \leq f_2 \leq \cdots$ and $f_j \in L^1([0,1])$ for each j, then $f \in L^1([0,1])$.
(D) If $f_1 \geq f_2 \geq \cdots \geq 0$ and $f_j \in L^1([0,1])$ for each j, then $f \in L^1([0,1])$.

(b) (3 points) Which of the following is true for the interval $[0,1]$ with respect to the Lebesgue measure?

(A) Every measurable set is Borel.
(B) For every Borel set B there exists an open set U containing B with $\mathcal{L}^1(U \setminus B) = 0$.
(C) For every measurable set A there exists a Borel set B containing A with $\mathcal{L}^1(B \setminus A) = 0$.
(D) An arbitrary union of measurable sets is measurable.

(c) (3 points) Which of the following implications is **false**?

(A) $f \in L^p(\mathbb{R})$ for every $p > 1 \quad \Longrightarrow \quad f \in L^1(\mathbb{R})$.
(B) $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad \Longrightarrow \quad f \in L^p(\mathbb{R})$ for every $1 < p < \infty$.
(C) $f \in L^3(\mathbb{R})$ and $g \in L^6(\mathbb{R}) \quad \Longrightarrow \quad fg \in L^2(\mathbb{R})$.
(D) $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R}) \quad \Longrightarrow \quad f * g \in L^1(\mathbb{R})$.

(d) (3 points) We consider the collection of sets $\mathcal{K} := \{[k, k+1) : k \in \mathbb{Z}\} \subset \mathcal{P}(\mathbb{R})$ and the function $\lambda : \mathcal{K} \to [0,\infty)$, $\lambda([k,k+1)) = 1$. Then we define

$$\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty], \qquad \mu(A) := \inf\left\{ \sum_j \lambda(K_j) : K_j \in \mathcal{K}, A \subset \bigcup_j K_j \right\}.$$

Which of the following properties is **not** true?

(A) μ is a Radon measure.
(B) The family of μ-measurable sets is a σ-algebra.
(C) Every μ-measurable function $f : \mathbb{R} \to \mathbb{R}$ takes at most countably many values.
(D) For every set $A \subset \mathbb{R}$, $\mathcal{L}^1(A) \leq \mu(A)$.

Question 2. (7 points)
Compute the following limits:

(a) (3 points)

$$\lim_{n\to\infty} \int_0^1 \frac{nx^{\frac{2}{3}}}{1+4n^2x^2}\, dx.$$

(b) (4 points)

$$\lim_{n\to\infty} \int_{[0,1]\times[0,1]} ny^{nx} e^{-x^2}\, d\mathcal{L}^2(x,y).$$

Hint: use the fact that $\int_0^1 t^a\, dt = \frac{1}{a+1}$ for $a \geq 0$ and reduce the two-dimensional integrals to integrals over an interval.

Question 3. (11 points)
Consider a set $\Omega \subset \mathbb{R}^n$ and a Radon measure μ on Ω. Let $f, f_k : \Omega \to \overline{\mathbb{R}}$, for $k = 1, 2, \ldots$, be μ-summable functions.

(a) (2 points) State what it means for the family $\{f_k\}$ to be uniformly μ-summable.

(b) (4 points) Show that if

$$\lim_{k\to\infty} \int_\Omega |f_k - f|\, d\mu = 0, \tag{$\star$}$$

then $f_k \xrightarrow{\mu} f$ (that is, the functions converge in measure) and $\{f_k\}$ is uniformly μ-summable.

(c) (2 points) Show by means of a counterexample that the converse is not true in general. That is, exhibit a sequence $\{f_k\}$ of μ-summable functions such that $f_k \xrightarrow{\mu} f$ and $\{f_k\}$ are uniformly μ-summable, but ($\star$) does not hold.

(d) (3 points) Suppose that for some $1 < p < \infty$ it holds that $\|f_k\|_{L^p(\Omega)} \leq 1$ for each k. Then show that the family $\{f_k\}$ is uniformly μ-summable.