Question 1 (10 points)

(a) [2 points] State the Dominated Convergence Theorem of Lebesgue.

(b) [4 points] Prove the Dominated Convergence Theorem.

(c) [4 points] Let μ be the Lebesgue measure on $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ be a μ-measurable set with $\mu(\Omega) < +\infty$, and $\{f_k\}$ be a sequence of functions $f_k \in L^1(\Omega, \mu)$ converging uniformly to f. Show that $f \in L^1(\Omega, \mu)$ and that

$$\lim_{k\to\infty} \int_\Omega f_k \, d\mu = \int_\Omega f \, d\mu.$$

Question 2 (12 points)

Compute the following limits:

(a) [4 points]

$$\lim_{n\to\infty} \int_{[0,n]} \left(1 + \frac{x}{n}\right)^n e^{-\pi x} \, dx.$$

(b) [4 points]

$$\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \sqrt{n \sin \frac{x}{n}} \, dx.$$

(c) [4 points]

$$\lim_{n\to\infty} \int_0^\infty \frac{\arctan\left(\frac{n}{x}\right)}{1+x^2} \, dx.$$

Question 3 (8 points)

In this question we work on $\mathbb{R}^n$ with the Lebesgue measure. Let $1 \le p < +\infty$, and consider the measurable functions $f_k, f : \mathbb{R}^n \to \overline{\mathbb{R}} = [-\infty, +\infty]$ for $k = 1, 2, 3, \ldots$, with $f \in L^p(\mathbb{R}^n)$.

(a) [4 points] Show that if $\|f_k - f\|_{L^p(\mathbb{R}^n)} \to 0$ as $k \to \infty$, then there exists a subsequence $\{f_{k_j}\}_{j\ge1}$ such that $f_{k_j} \to f$ as $j \to \infty$ almost everywhere.

(b) [4 points] Show, by means of a counterexample, that in general convergence in L^p does not imply that the full sequence converges almost everywhere.