Question 1 — Multiple Choice

[12 Points]

MC1 (i) [3 Points] Let μ be a Radon measure on $\mathbb{R}^n$ and suppose that $A \subset B_1(0)$ is **not** μ-measurable. Which of the following is true?

(A) Necessarily $\mu(A) = 0$.

(B) Necessarily $0 < \mu(A) < +\infty$.

(C) Necessarily $\mu(A) = +\infty$.

(D) It depends on the particular set A and the measure μ.

MC1 (ii) [3 Points] Let $(E_j)_{j=1}^{\infty}$ be measurable subsets of $\mathbb{R}$ such that $\mathcal{L}^1(E_1) < +\infty$ and $E_1 \supset E_2 \supset \cdots$. Let $E := \bigcap_{j=1}^{\infty} E_j$. Which of the following is **not** true in general?

(A) $\chi_{E_j} \to \chi_E$ in L^1.

(B) $\chi_{E_j} \to \chi_E$ in L^p for all $1 < p < +\infty$.

(C) $\chi_{E_j} \to \chi_E$ in L^∞.

(D) $\chi_{E_j} \to \chi_E$ almost everywhere.

MC1 (iii) [3 Points] Let $f_1, f_2, \ldots \in L^1(\mathbb{R})$ and suppose that $f_j \geq 0$ almost everywhere and $\|f_j\|_{L^1} \to 0$ as $j \to \infty$. Consider the following statements:

(i) $\displaystyle\sum_{j=1}^{\infty} f_j \in L^1(\mathbb{R})$ (ii) $\displaystyle\limsup_{j\to\infty} f_j = 0$ a.e. (iii) $\displaystyle\liminf_{j\to\infty} f_j = 0$ a.e.

Which of the above are always true?

(A) All of them.

(B) Only (ii) and (iii).

(C) Only (iii).

(D) None of them.

MC1 (iv) [3 Points] Fix an integer $n \geq 2$. What is set of all exponents $p \in [1, +\infty]$ such that for every $f \in L^p(\mathbb{R}^n, \mathcal{L}^n)$ it holds that the integral

$$\int_{B_1(0)} |f(x)| \frac{1}{|x|} \, \mathrm{d}\mathcal{L}^n(x)$$

is finite?

(A) $(1, +\infty]$

(B) $\left(\frac{n}{n-1}, +\infty\right]$

(C) $(n, +\infty]$

(D) $\varnothing$

Question 2

[10 Points]

Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$ be a μ-measurable set.

Q2 (i) [2 Points] Give the definition of the space $L^p(\Omega, \mu)$ for $1 \leq p \leq +\infty$.

Q2 (ii) [4 Points] Prove that, assuming that $\mu(\Omega) < +\infty$, we have the inclusion $L^q(\Omega, \mu) \subset L^p(\Omega, \mu)$ whenever $1 \leq p < q \leq +\infty$.

Q2 (iii) [4 Points] Show that in the case of $\mu = \mathcal{L}^1$ and $\Omega = [0, 1] \subset \mathbb{R}$, the inclusion is strict for every $1 < p < q < +\infty$.

Question 3

[8 Points]

Q3 (i) [4 Points] Compute the limit

$$\lim_{n\to\infty} \int_0^\infty \frac{1}{1 + x^{1+\frac{1}{n}}} \, \mathrm{d}x$$

Q3 (ii) [4 Points] Compute the limit

$$\lim_{n\to\infty} \int_0^\pi \left(1 + \frac{\log x}{n}\right)^n \cos x \, \mathrm{d}x.$$