Exercise 1. (9 points)
For all of these questions there is **one** correct answer. For each part, 3 points are awarded if correct, 0 points if not.

(a) Let $E \subset \mathbb{R}$ be a set with $\mathcal{L}^1(E) = 0$. Which of the following statements is correct?

(A) E must be at most countable.
(B) E must be Lebesgue-measurable.
(C) E must be bounded.
(D) E must be an element of the Borel σ-algebra.

(b) The value of the limit

$$\lim_{n\to\infty} \int_0^{1/n} \frac{e^{nx}}{\sqrt{x}}\, dx$$

is

(A) 0. (B) 1. (C) 2. (D) ∞.

(c) Let $\{f_j\}$ be a sequence of real-valued $\mathcal{L}^1$-measurable functions on $[0, 1]$ converging pointwise $\mathcal{L}^1$-almost everywhere to a function $f : [0, 1] \to \mathbb{R}$. Which one of the following assertions is FALSE?

(A) If $|f_j| \leq K$ for some $K > 0$, then $f_j \to f$ in L^1.
(B) There is a set $A \subset [0, 1]$ with $\mathcal{L}^1(A) = 1/2$ such that $f_j \to f$ uniformly in A.
(C) If all f_j are in L^1, then $f_j \to f$ in L^1.
(D) $f_j \to f$ in measure.

Exercise 2. (10 points)
The goal of this exercise is to construct a subset of $[0, 1)$ which is not $\mathcal{L}^1$-measurable. We will use the binary operation $\oplus : [0, 1) \times [0, 1) \to [0, 1)$, defined by

$$x \oplus y = \begin{cases} x + y, & \text{if } x + y < 1 \\ x + y - 1, & \text{if } x + y \geq 1. \end{cases}$$

Moreover we define $E \oplus x := \{a \oplus x : a \in E\}$ for $E \subseteq [0, 1)$ and $x \in [0, 1)$.

(a) (4p) Show that if E is $\mathcal{L}^1$-measurable, then so is $E \oplus x$ for any $x \in [0, 1)$ and $\mathcal{L}^1(E) = \mathcal{L}^1(E \oplus x)$.

(b) (2p) Apply the axiom of choice in order to construct a set $P \subseteq [0, 1)$ with the property that, for any real number z, there exists exactly one rational number $r \in \mathbb{Q}$ such that $z - r \in P$.

(c) (4p) Show that any set P with this property is not $\mathcal{L}^1$-measurable.

Exercise 3. (11 points)
(a) (4p) For two functions $f, g \in L^1(\mathbb{R}^n, dx)$, define $f \star g$ and show that

$$\|f \star g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}.$$

For each $r > 0$, let $\rho_r := \mathcal{L}^n(B_r)^{-1} \chi_{B_r}$, where $B_r = B(0, r) \subset \mathbb{R}^n$.

(b) (3p) Show that if $g : \mathbb{R}^n \to \mathbb{R}$ is continuous and compactly supported, then $\|g \star \rho_r - g\|_{L^1} \to 0$ as $r \to 0$.

(c) (4p) Show that, for any $f \in L^1(\mathbb{R}^n)$, there is a sequence $r_k \to 0$ such that

$$f \star \rho_{r_j}(z) \xrightarrow{j \to \infty} f(z)$$

for $\mathcal{L}^n$-a.e. $z \in \mathbb{R}^n$. **Hint:** you may use the fact that continuous and compactly supported functions are dense in $L^1(\mathbb{R}^n)$.