The written exam of Analysis III (Measure Theory) will consist of three exercises. The exercises will contain between 2 and 4 questions. In some questions you may be asked to state and/or to prove results that have been presented during the lectures or to prove something inspired by what you have already seen in the exercise sheets. Some of the exercises below will be solved during the lecture of 23rd December 2022.

Exercise 1.

(a) State and prove the *Dominated Convergence Theorem.*

(b) Let μ be a Radon measure on $\mathbb{R}^n$. Let $\Omega \subseteq \mathbb{R}^n$ be μ-measurable with $\mu(\Omega) < +\infty$ and $f_k \in L^1(\Omega, \mu)$ be uniformly convergent to f. Show that $f \in L^1(\Omega, \mu)$ and $\int_\Omega f_k d\mu \to \int_\Omega f d\mu$ as $k \to +\infty$.

Exercise 2.
Let $\mathcal{L}^n$ be the Lebesgue measure on $\mathbb{R}^n$ and $f, g \in L^1(\mathbb{R}^n, \mathcal{L}^n)$.

(a) Show that the function $h : \mathbb{R}^{2n} \to \mathbb{R}$ defined by $h(x, y) = f(x)g(y)$ is $\mathcal{L}^{2n}$-measurable.

(b) Show that $h \in L^1(\mathbb{R}^{2n}, \mathcal{L}^{2n})$ and

$$\int_{\mathbb{R}^{2n}} h(x, y) d\mathcal{L}^{2n} = \int_{\mathbb{R}^n} f(x) d\mathcal{L}^n \int_{\mathbb{R}^n} g(y) d\mathcal{L}^n.$$

Exercise 3.
Let μ be a Radon measure and $\Omega \subseteq \mathbb{R}^n$ be μ-measurable.

(a) State Vitali's Theorem.

(b) Show that the hypothesis $\mu(\Omega) < +\infty$ in Vitali's Theorem is necessary.

(c) Show that in the case $\mu(\Omega) < +\infty$, then Lebesgue's Dominated Convergence Theorem implies Vitali's Theorem.

Exercise 4.
Let $1 \le p < +\infty$, f_k, $f \colon \mathbb{R}^n \to [-\infty, +\infty]$.

(a) Show that if $\|f_k - f\|_{L^p} \to 0$ as $k \to +\infty$, then there is a subsequence with $f_{k_j} \to f$ as $j \to +\infty$ μ-a.e.

(b) Show that in general convergence in L^p does not imply that the full sequence converges μ-a.e.

Exercise 5.
Let $f(x, y) = \frac{\sin(x)}{x} e^{-xy}$.

(a) Verify if $f \in L^1((0, 1] \times [0, +\infty))$.

(b) Verify if $f \in L^1((0, +\infty) \times [1, +\infty))$.

Exercise 6.

(a) Describe the Cantor Dust.

(b) Show that its Hausdorff dimension is 1.

Exercise 7.

(a) Describe the construction of the Vitali set.

(b) Show that the Vitali set is not Lebesgue-measurable.

(c) Show that for every set $A \subset (0,1)$ with $\mathcal{L}^1(A) > 0$, there exists $B \subset A$ which is not $\mathcal{L}^1$ measurable.

(d) Could you give an example of a sequence of disjoint sets which are not Lebesgue measurable for which the σ-additivity of the Lebesgue measure does not hold?

Exercise 8.
Let $f\colon \mathbb{R}^n \to [0,+\infty]$ be $\mathcal{L}^n$-measurable. We set

$$\lambda(E) = \int_E f d\mathcal{L}^n.$$

(a) Show that λ is a measure on $\mathbb{R}^n$.

(b) Show that

$$\int_{\mathbb{R}^n} g d\lambda = \int_{\mathbb{R}^n} f g d\mathcal{L}^n$$

for every $g\colon \mathbb{R}^n \to [0,+\infty]$ λ-summable.

(c) Show that $f(x) = x^{-\alpha}$ is summable on $(0,1)$ for all $\alpha \in (0,1)$.

(d) Compute $\int_0^y f(x)dx$.

Exercise 9.
Let $\mu\colon \mathcal{P}(X) \to [0,+\infty]$ be a measure and let A_n be a sequence of sets in $\mathcal{P}(X)$.

(a) Prove that

$$\mu(\liminf_{n\to+\infty} A_n) \le \liminf_{n\to+\infty} \mu(A_n).$$

(b) Assume that for some $m \in \mathbb{N}$ we have $\mu(\cup_{n\ge m} A_n) < +\infty$. Then show that

$$\limsup_{n\to+\infty} \mu(A_n) \le \mu(\limsup_{n\to+\infty} A_n).$$

(c) Prove that if $\sum_{n=0}^{\infty} \mu(A_n) < +\infty$, then $\mu(\limsup_{n\to+\infty} A_n) = 0$

Exercise 10.
Define $I_1 = [0,1]$. For every $n \ge 1$, let $I_{n+1} \subset I_n$ be the collection of intervals obtained by removing from every interval in I_n its centered open subinterval of length $(1/3)^n$. Then define by $C_{1/3} = \bigcap_{n=1}^{\infty} I_n$, the *Cantor set*.

Show that:

(a) $C_{1/3}$ is Lebesgue-measurable with measure $\mathcal{L}^1(C_{1/3}) = 0$.

(b) $C_{1/3}$ is uncountable.

Exercise 11.
Let $\mathcal{L}^n$ be the Lebesgue measure in $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ be $\mathcal{L}^n$-measurable and $f_k, f : \Omega \to \mathbb{R}$ be $\mathcal{L}^n$-measurable functions. Let $\varphi \colon \mathbb{R} \to \mathbb{R}$.

(a) Show that if φ is continuous and $f_k \to f$ $\mathcal{L}^n$-a.e, then $\varphi \circ f_k \to \varphi \circ f$ $\mathcal{L}^n$-a.e as well.

(b) Give the definition of convergence in measure and show the link between convergence in measure and convergence almost everywhere.

(c) Show that if φ is uniformly continuous and $f_k \to f$ in measure then $\varphi \circ f_k \to \varphi \circ f$ in measure as well.

(d) Provide an example showing that part (c) does not hold if φ is continuous but not uniformly continuous.

Exercise 12.

(a) State Fubini and Tonelli's theorems.

(b) Show that $f \colon [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{(x-1/2)^3} & \text{if } 0 < y < |x - 1/2| \\ 0 & \text{otherwise} \end{cases}$$

is not $\mathcal{L}^2$ summable.

Exercise 13.

(a) State Fubini and Tonelli's theorems.

(b) If f is Lebesgue summable on $(0,a)$ and $g(x) = \int_x^a \frac{f(t)}{t} dt$ then g is Lebesgue summable on $(0,a)$ and

$$\int_0^a g(x)dx = \int_0^a f(t)dt.$$

Exercise 14.
Let

$$f(x) = \begin{cases} x^{-1/2} & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

and let $(r_n)_n$ be an enumeration of rational numbers. Then define $f_n(x) = 2^{-n} f(x - r_n)$ and

$$g(x) = \sum_{n=1}^{\infty} f_n(x).$$

(a) State Beppo Levi's theorem.

(b) Apply Beppo Levi's theorem to show that

$$\sum_{m=1}^{\infty} \int_{\mathbb{R}} f_n d\mathcal{L}^1 = \int_{\mathbb{R}} \left(\sum_{m=1}^{\infty} f_n \right) d\mathcal{L}^1.$$

(c) Show that g is Lebesgue summable and $g < +\infty$ almost everywhere.

(d) Show that $g^2 < +\infty$ $\mathcal{L}^1$-a.e but g^2 is not Lebesgue summable.

Exercise 15.

(a) Give the definition of s-dimensional Hausdorff measure in $\mathbb{R}^n$.

(b) Show it is a Borel-regular measure.

(c) Show it is not a Radon measure for $0 \leq s < n$.

Exercise 16.

(a) Give the definition of s-dimensional Hausdorff measure.

(b) Show that $\mathcal{H}^0$ coincides with the counting measure.

(c) Show that

$$\dim_{\mathcal{H}}(A) := \inf\{s \geq 0 : \quad \mathcal{H}^s(A) = 0\} = \sup\{s \geq 0 : \quad \mathcal{H}^s(A) = +\infty\}.$$

Exercise 17.

(a) Give the definition of an algebra and a σ-algebra and mention some concrete examples.

(b) Provide examples of algebras which are not σ-algebras.

Exercise 18.

(a) State Fatou's Lemma.

(b) Prove Fatou's Lemma.

(c) Is the assumption in Fatou's Lemma that $f_n \geq 0$ necessary?

Exercise 19.

(a) In an uncountable set X, consider the class

$$\mathcal{E} = \{A \in \mathcal{P}(X) \mid A \text{ is countable or } A^c \text{ is countable}\}.$$

Show $\mathcal{E}$ is a σ-algebra (here "countable" stands for "at most countable").

(b) Show that $\mathcal{E}$ is a σ-algebra which is strictly smaller than $\mathcal{P}(X)$.

Exercise 20.

(a) Explain what it means for μ to be an *additive* and a *σ-additive* set function and show some examples.

(b) Could you show that an additive function is σ-additive if and only if it is σ-subadditive?

Exercise 21.

(a) State the Carathéodory criterion of measurability.

(b) Show that the set

$$\Sigma = \{A \subseteq X\colon\ A \text{ is } \mu\text{-measurable}\}$$

is a σ-algebra.

Exercise 22.

(a) Give the definition of Lebesgue measure in $\mathbb{R}^n$.

(b) Prove the following regularity result: for every $A \subseteq \mathbb{R}^n$ it holds

$$\mathcal{L}^n(A) = \inf_{A\subseteq G} \mathcal{L}^n(G), \quad G \text{ open}.$$

(c) Show that if A is $\mathcal{L}^n$-measurable, then for all $\varepsilon > 0$ there exists $G \supseteq A$ open such that $\mathcal{L}^n(G \setminus A) < \varepsilon$.

Exercise 23.
Let μ be a Radon measure on Ω and f_n, f functions in $L^p(\Omega,\mu)$ $(1 \le p < +\infty)$ such that $f_n \to f$ in L^p norm.

(a) Show that there exists a subsequence $\{f_{k_n}\}$ and a function $g \in L^p(\Omega,\mu)$ such that $|f_{k_n}| \le g$. That is, there is a dominating function once we pass to an appropriate subsequence.
Hint: choose a subsequence with $\|f_{k_n} - f\|_{L^p} \le 2^{-n}$.

(b) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\varphi(t) \le |t|^p\ \forall t \in \mathbb{R}$. Show that then

$$\lim_{n\to\infty} \int_\Omega \varphi(f_n(x))\, d\mu(x) = \int_\Omega \varphi(f(x))\, d\mu(x).$$

Hint: it is enough to show this limit along an appropriate subsequence (why?).