Exercise 2.1.
(a) For every $A \in \mathcal{P}(X)$ we define

$$\mu(A) := \begin{cases} \#A & \text{if } A \text{ is finite,} \\ +\infty & \text{otherwise.} \end{cases}$$

Prove that μ is a measure on $\mathcal{P}(X)$ and that every A is $\mu-$measurable. The measure μ is called the **counting measure**.

(b) Let $\mu : \mathcal{P}(X) \to [0, +\infty]$, $\mu(\emptyset) = 0, \mu(A) = 1, A \neq \emptyset$. Show that μ is a measure and that $A \subseteq X$ is μ-measurable $\iff$ $A = \emptyset$ or $A = X$.

Exercise 2.2.
Given a measure μ on a set X, we define the set of atoms of μ as

$$A_\mu := \{x \in X : \{x\} \text{ is measurable and } \mu(\{x\}) > 0\}.$$

(a) Assuming that $\mu(X) < +\infty$, show that A_μ is at most countable.

(b) Is the same true if μ is only assumed to be σ-finite? And in general? Show it or give a counterexample.

(c) Construct an example of measure μ on an uncountable set X such that $\mu(\{x\}) > 0$ for every $x \in X$ but $\mu(X) < \infty$. This shows that the condition of the measurability of $\{x\}$ in the definition of A_μ cannot be removed.

Exercise 2.3. (Upper and lower semicontinuity of measures.)
Let $\mathcal{E}$ be a σ-algebra on a set X and $\mu : \mathcal{E} \to [0, \infty]$ a σ-additive function on $\mathcal{E}$. For a sequence $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{E}$,
(a) show that

$$\mu\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mu(A_n).$$

(b) show that also

$$\limsup_{n\to\infty} \mu(A_n) \leq \mu\left(\limsup_{n\to\infty} A_n\right)$$

holds provided that $\mu(X) < \infty$.

Exercise 2.4. ♣
Let $\mathcal{E}$ be a σ-algebra on a set X and $\mu : \mathcal{E} \to [0, \infty]$ a σ-additive function on $\mathcal{E}$. Which of the following are true for an arbitrary sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{E}$?

(a)
$$\mu\left(\liminf_{n\to\infty} A_n\right) = \limsup_{n\to\infty} \mu(A_n).$$

(b)
$$\liminf_{n\to\infty} A_n^c = \left(\limsup_{n\to\infty} A_n\right)^c.$$

(c) Whenever $B \subseteq \bigcup_{n=1}^{\infty} A_n$,
$$\mu(B) < \sum_{n=1}^{\infty} \mu(A_n).$$

(d) Whenever $B \subseteq \bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$,
$$\mu(B) = \sum_{n=1}^{\infty} \mu(A_n).$$