Exercise 3.1.
Denote by λ the Lebesgue measure on $\mathbb{R}$. Let $E \subset [0,1]$ be a Lebesgue measurable set of strictly positive measure, i.e. $\lambda(E) > 0$. Show that for any $0 \leq \delta \leq \lambda(E)$, there exists a measurable subset of E having measure exactly δ.
Hint: Consider the function $[0,1] \ni t \mapsto \lambda([0,t] \cap E)$.

Exercise 3.2.
Let $\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ be the function

$$\mu(A) := \sqrt{\lambda(A)}$$

for $A \subseteq \mathbb{R}$, where λ denotes the Lebesgue measure.
(a) Show that μ is a measure.

(b) ★ What is the σ-algebra of μ-measurable sets?

Exercise 3.3.
Recall that the system of elementary sets is defined as

$$\mathcal{A} := \{A \subset \mathbb{R}^n \mid A \text{ is the union of finitely many disjoint intervals}\}.$$

(a) Prove that $\mathcal{A}$ is an algebra. To simplify the notation you may assume that $n = 1$.

(b) ★ Show that the volume function vol introduced in the lecture[1] for elementary sets is a pre-measure.

Remark: For $I = I_1 \times \ldots \times I_n$ an interval in $\mathbb{R}^n$, its volume is defined by

$$\mathrm{vol}(I) = \prod_{k=1}^{n} \mathrm{vol}(I_k),$$

where for an interval $I_k \subseteq \mathbb{R}$, $\mathrm{vol}(I_k)$ is the length of I_k.

[1] Definition 1.3.1 in the Lecture Notes.

Exercise 3.4.
Define $\mathcal{A}$ to be the algebra in $\mathbb{R}$ generated by the half-closed intervals of the form $[a, b)$ for every $-\infty \leq a \leq b \leq \infty$. Note that any element in $\mathcal{A}$ can be expressed as the disjoint union of finitely many intervals of the type described before. Moreover, we define:

$$\eta : \mathcal{A} \to [0, +\infty], \quad \eta(A) := \begin{cases} +\infty & \text{if } A \neq \emptyset, \\ 0 & \text{else.} \end{cases}$$

(a) Check that η is a pre-measure. Find the Carathéodory-Hahn extension μ of η and the σ-algebra Σ of μ-measurable sets.

(b) Show that there is an extention $\tilde{\mu} : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ of η such that $\tilde{\mu}|_\Sigma \neq \mu$.

(c) Why is not the existence of $\overline{\mu}$ in contradiction with the uniqueness statement of Theorem 1.2.21 in the Lecture Notes?

Exercise 3.5. ★
Let $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ be a measure with the following property: there is a real number $s > n$ such that for every $x \in \mathbb{R}^n$ and $r > 0$,

$$\mu(B(x, r)) \leq r^s.$$

Show that $\mu \equiv 0$. Here $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ denotes the open ball with center x and radius r in $\mathbb{R}^n$.

Exercise 3.6. ♣
Let μ be a measure on $\mathbb{R}^n$ and $A, B_1, B_2, \ldots \subset \mathbb{R}^n$ be such that $A \subseteq \liminf_{k\to\infty} B_k$ and $\sum_{k=1}^{\infty} \mu(B_k) < \infty$. Which of the following statements are true?
(a) $\mu(A) > 0$.

(b) $\mu(A) = 0$.

(c) A is a measurable set for any choice of μ and $B_1, B_2, \ldots \subset \mathbb{R}^n$.

(d) Every point of A belongs to infinitely many of the B_k.