

Exercise 6.1.

Let μ be a Borel measure. Show that μ is metric.

Exercise 6.2.

Let $f : [0, 1] \rightarrow \mathbb{R}$ be an α -Hölder continuous function, namely, there is a constant $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|^\alpha$ for every $x, y \in [0, 1]$, where $0 < \alpha \leq 1$ is a fixed number. Let $G = \{(x, f(x)) : x \in [0, 1]\} \subset \mathbb{R}^2$ denote its graph.

(a) Show that $\mathcal{L}^2(G) = 0$.

(b) ★ Show moreover that $\mathcal{H}^s(G) = 0$ for every $s > 2 - \alpha$.

Exercise 6.3.

Show that the graph of the function $g : [0, 1] \rightarrow \mathbb{R}$ defined by

$$g(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

has Lebesgue measure zero in \mathbb{R}^2 .

Exercise 6.4.

For $s \geq 0$ and $\emptyset \neq A \subset \mathbb{R}^n$, we define

$$\mathcal{H}_\infty^s(A) := \inf \left\{ \sum_{k \in I} r_k^s : A \subset \bigcup_{k \in I} B(x_k, r_k), r_k > 0 \right\},$$

where the set of indices I is at most countable. One can check that \mathcal{H}_∞^s is a measure. Prove that $\mathcal{H}_\infty^{1/2}$ is not Borel on \mathbb{R} .

Remark. Note that the definition of \mathcal{H}_∞^s coincides with Definition 1.8.1 in the Lecture Notes for $\delta = \infty$.

Exercise 6.5. ♣

Recall that a measure is called Radon if it is Borel regular and finite on compact sets. Which of the following statements are true?

- (a) The Dirac measure δ_0 is a Radon measure on \mathbb{R} .
- (b) The Lebesgue measure \mathcal{L}^n on \mathbb{R}^n is a Radon measure.
- (c) For any nondecreasing and left-continuous function $F : \mathbb{R} \rightarrow \mathbb{R}$, the Lebesgue–Stieltjes measure Λ_F is **not** a Radon measure on \mathbb{R} .
- (d) For any $s > 0$, the Hausdorff measure \mathcal{H}^s is a Radon measure on \mathbb{R}^n .
- (e) For every set $A \subset \mathbb{R}^n$, it holds $\mathcal{H}^{n+1}(A) = +\infty$.

Exercise 6.6.

Consider the continuous function $f : [0, 1] \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$$

and its graph

$$A := \{(x, f(x)) \mid x \in [0, 1]\} \subset \mathbb{R}^2.$$

- (a) ★ Show that $\mathcal{H}^1(A) = \infty$.

Hint: use the σ -additivity of \mathcal{H}^1 on Borel sets to decompose the curve A into pieces which “look like” straight lines and try to compare their \mathcal{H}^1 -measure with their length.

- (b) Show that $\mathcal{H}^s(A) = 0$ if $s > 1$.
- (c) Conclude that $\dim_{\mathcal{H}}(A) = 1$.