Exercise 7.1. ♣
Let μ be a measure on $\mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$ a μ-measurable set. Which of the following statements are true?
(a) If $f : \Omega \to \mathbb{R}$ is μ-measurable and $g : \mathbb{R} \to \mathbb{R}$ is a Borel function[1], then $g \circ f$ is **not** μ-measurable.

(b) There exists a μ-non-measurable function $f \geq 0$ such that $\sqrt{f}$ is μ-measurable.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is continuous $\mathcal{L}^1$-almost everywhere then f is $\mathcal{L}^1$-measurable.

(d) If $(f_k)_{k\in\mathbb{N}}$ is a sequence of μ-measurable functions $f_k : \mathbb{R}^n \to \mathbb{R}$, then the set

$$E := \{x \in \mathbb{R}^n : \lim_{k\to\infty} f_k(x) \text{ exists and is finite}\}$$

is μ-measurable.

(e) Let $f : [0,1] \to \mathbb{R}$ and suppose that for every $c \in \mathbb{R}$, the set $\{x \in [0,1] : f(x) = c\}$ is $\mathcal{L}^1$-measurable. Then f is $\mathcal{L}^1$-measurable.

(f) If $f : \Omega \to \mathbb{R}$ is differentiable and μ-measurable, then f' is μ-measurable.

Exercise 7.2.
Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Show that the following statements are equivalent.

(i) $f^{-1}(U)$ is μ-measurable for every open set $U \subset \mathbb{R}$.

(ii) $f^{-1}(B)$ is μ-measurable for every Borel set $B \subset \mathbb{R}$.

(iii) $f^{-1}((-\infty, a))$ is μ-measurable for every $a \in \mathbb{R}$.

Exercise 7.3.
Let (X, μ, Σ) be a measure space and $f, g : X \to \mathbb{R}$ two measurable functions on X. Show that the sets $\{x \mid f(x) = g(x)\}$ and $\{x \mid f(x) < g(x)\}$ are measurable.

[1] A function $g : \mathbb{R} \to \mathbb{R}$ is Borel iff $g^{-1}(U)$ is a Borel set for every open set $U \subseteq \mathbb{R}$.

Exercise 7.4. ★

In this exercise, we construct a set which is Lebesgue measurable but not Borel, and use the construction to give an example of a continuous $G : \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable function $H : \mathbb{R} \to \mathbb{R}$ such that $H \circ G$ is not Lebesgue measurable.
(a) Let $h : [0, 1] \to [0, 1]$ be the Cantor function, which is the unique monotonically increasing extension of the function $F : C \to [0, 1]$ seen in Exercise 5.2, where $C \subset [0, 1]$ is the Cantor set. Define $g : [0, 1] \to [0, 2]$ by $g(x) := h(x) + x$. Show that g is strictly monotone and a homeomorphism.

(b) Show that $\mathcal{L}^1(g(C)) = 1$.

Hint: Use the natural decomposition of $[0, 1] \setminus C$ to deduce the result.

(c) Use Exercise 4.3 (a) to find a non-measurable subset $E \subset g(C)$ and define $A := g^{-1}(E)$. Show that A is a Lebesgue zero set and thus Lebesgue measurable.

(d) Show that A is not a Borel set.

Hint: Otherwise, the preimage of A with respect to continuous maps would necessarily be Borel and thus Lebesgue measurable as well.

(e) Find appropriate H, G as outlined above such that $H \circ G$ is not Lebesgue measurable, using the sets and functions introduced in the previous subtasks.

Exercise 7.5.

Let μ be a Borel measure on $\mathbb{R}$. Show that every monotone function $f\colon [a, b] \to \mathbb{R}$ is μ-measurable.