Exercise 8.1.
Let μ be a measure on $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ a μ-measurable set and $f : \Omega \to [0, \infty]$ a μ-measurable function. Consider the sets $A_j \subseteq \Omega$ from Theorem 2.2.6 of the Lecture Notes, defined so that the sequence of functions

$$f_k = \sum_{j=1}^{k} \frac{1}{j} \chi_{A_j}$$

converges pointwise to f. Show that if f is bounded, then f_k converge uniformly to f, that is,

$$\sup_{x \in \Omega} |f(x) - f_k(x)| \longrightarrow 0 \text{ as } k \to \infty.$$

Exercise 8.2.
Let $f : \Omega \to \overline{\mathbb{R}}$ be μ-measurable. Prove that there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of simple functions $f_k : \Omega \to \mathbb{R}$ satisfying $|f_k(x)| \leq |f_{k+1}(x)|$ and $\lim_{k\to\infty} f_k(x) = f(x)$ for every $x \in \Omega$.
In particular, this shows that $|f_k(x)| \leq |f(x)|$ for each $x \in \Omega$ and $k \geq 1$.

Exercise 8.3. ♣
Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of Lebesgue-measurable functions on $\mathbb{R}$. Which of the following assertions are true?
(a) If $(f_k)_{k\in\mathbb{N}}$ converges to 0 in measure, there is a subset $A \subset \mathbb{R}$ of positive measure and a subsequence which converges uniformly on A.

(b) $\{x \in \mathbb{R} : f_k(x) \to \infty\}$ is non-measurable.

(c) $\{x \in \mathbb{R} : \lim f_k(x) \text{ exists in } \mathbb{R}\}$ is measurable.

(d) If f is a non-measurable function, then $|f|$ is non-measurable.

(e) The sequence $g_n(x) = e^{-n(1-\sin x)}$ converges in measure to the function $g \equiv 0$ on any bounded interval $[a, b] \subset \mathbb{R}$.

Exercise 8.4.
Let $f_k \colon \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-measurable functions, for $k \in \mathbb{N}$. Assume that

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \mid |f_k(x) - f_{k+1}(x)| > 2^{-k}\}) < 2^{-k}$$

for all $k \in \mathbb{N}$. Show that the limit $\lim_{k\to\infty} f_k(x)$ exists almost everywhere.

Exercise 8.5.
Let μ be a measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ be μ-measurable. Let $f\colon \Omega \to \overline{\mathbb{R}}$ be a finite, μ-measurable function, and $(f_k)_{k\in\mathbb{N}}$ a sequence of μ-measurable functions $f_k\colon \Omega \to \overline{\mathbb{R}}$.
(a) Suppose that every subsequence $(f_{k_j})_{j\in\mathbb{N}}$ contains a subsequence that converges to f in measure. Show that the whole sequence $(f_k)_{k\in\mathbb{N}}$ converges to f in measure.

(b) Show that the analogous statement from (a) is not true if we replace "convergence in measure" by "convergence pointwise almost everywhere". Namely, show that there exists a sequence (f_k) and a function f such that every subsequence of (f_k) has a further subsequence that converges a.e. to f, but the whole (f_k) does not converge a.e. to any function.

Exercise 8.6. ★
Counterexample to $\varepsilon = 0$ in Lusin's Theorem: Find an example of a $\mathcal{L}^1$-measurable function $f : [0,1] \to \mathbb{R}$ such that for every $\mathcal{L}^1$-measurable set $M \subset [0,1]$ with $\mathcal{L}^1(M) = 1$, the restriction $f|_M : M \to \mathbb{R}$ is discontinuous in all but finitely many points of M.
Hint: You may use that there exists a Lebesgue measurable subset $A \subset [0,1]$ such that

$$\mathcal{L}^1(U \cap A) \cdot \mathcal{L}^1(U \cap A^c) > 0$$

for all nonempty open subsets $U \subset [0,1]$. Such a set A can be constructed using the fat Cantor set (see Exercise 1.6.2 in the lecture notes).

Exercise 8.7.
Counterexample to $\delta = 0$ in Egoroff's Theorem: Find an example of a sequence of $\mathcal{L}^1$-measurable functions $f_k : [0,1] \to \overline{\mathbb{R}}$ that converges pointwise almost everywhere to a $\mathcal{L}^1$-measurable ($\mathcal{L}^1$-almost everywhere finite) function $f : [0,1] \to \overline{\mathbb{R}}$, but for every set $F \subseteq [0,1]$ with $\mathcal{L}^1(F) = \mathcal{L}^1([0,1])$ the convergence on F is not uniform.