Exercise 9.1. ♣
Which of the following statements are true?
5 correct answers are enough for the bonus.
(a) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions on $\mathbb{R}$ such that $f_n \to f$ almost everywhere. Then $\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)\,dx$ exists and

$$\int_{\mathbb{R}} f(x)\,dx \leq \lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)\,dx.$$

(b) There exists a sequence of Lebesgue measurable functions $(f_n)_{n\in\mathbb{N}}$ such that f_n converges to 0 in measure on $\mathbb{R}$ but no subsequence converges uniformly on any subset of positive measure.

(c) Let $f \in L^1([0,1])$.[a] Then for each nonnegative integer n, $x^n f(x) \in L^1([0,1])$.

(d) Let $f \in L^1((0,+\infty))$. Then $\lim_{x\to+\infty}|f(x)| = 0$.

(e) Let $f \in L^1((0,+\infty))$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ sucht that $x_n \to \infty$ and $\lim_{n\to\infty} x_n f(x_n) = 0$.

(f) There exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^1((0,+\infty))$ such that $|f_n(x)| \leq 1$ for all x and all n, $\lim_{n\to\infty} f_n(x) = 0$ for all x, and $\lim_{n\to\infty}\int_{(0,\infty)} f_n(x)\,dx = 1$.

(g) There exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^1([0,1])$ such that $f_n \to 0$ pointwise and yet $\int_{[0,1]} f_n(x)\,dx \to +\infty$.

[a] $L^1(A) := \{f : A \to \mathbb{R} : \int_A |f(x)|dx < \infty\}$, for any $A \subseteq \mathbb{R}$.

Exercise 9.2.
(a) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of μ-measurable functions on a μ-measurable set $\Omega \subset \mathbb{R}^n$. Show that the series $\sum_{k=1}^{\infty} f_k(x)$ converges μ-almost everywhere, if

$$\sum_{k=1}^{\infty}\int_{\Omega}|f_k|d\mu < \infty.$$

(b) Let $\{r_k\}_{k\in\mathbb{N}}$ be an ordering of $\mathbb{Q}\cap[0,1]$ and $(a_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be such that $\sum_{k=1}^{\infty} a_k$ is absolutely convergent. Show that $\sum_{k=1}^{\infty} a_k|x-r_k|^{-1/2}$ is absolutely convergent for almost every $x \in [0,1]$ (with respect to the Lebesgue measure).

Exercise 9.3.
Find an example of a continuous bounded function $f\colon [0,\infty) \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$ and
$$\int_0^\infty |f(x)|^p dx = \infty \ ,$$
for all $p > 0$.

Exercise 9.4.
Let $f, g : \Omega \to \overline{\mathbb{R}}$ be μ-summable functions and assume that
$$\int_A f d\mu \leq \int_A g d\mu$$
for all μ-measurable subsets $A \subset \Omega$. Show that $f \leq g$ μ-almost everywhere. Moreover, conclude that, if
$$\int_A f d\mu = \int_A g d\mu$$
for all μ-measurable subsets $A \subset \Omega$, then $f = g$ μ-almost everywhere.

Exercise 9.5.
Let $f_n\colon \mathbb{R} \to \overline{\mathbb{R}}$ be Lebesgue measurable functions. Find examples for the following statements.
(a) $f_n \to 0$ uniformly, but not $\int |f_n| dx \to 0$.

(b) $f_n \to 0$ pointwise and in measure, but neither $f_n \to 0$ uniformly nor $\int |f_n| dx \to 0$.

(c) $f_n \to 0$ pointwise, but not in measure.

Exercise 9.6.
Let $f : [0,1] \to \mathbb{R}$ be $\mathcal{L}^1$-summable. Show that for a set $E \subset [0,1]$ of positive measure it holds that
$$f(x) \leq \int_{[0,1]} f(y)\, d\mathcal{L}^1(y)$$
for every $x \in E$.