Exercise 10.1.
Show that a μ-measurable function $f\colon \Omega \to \mathbb{R}$ is integrable if and only if either $\int_\Omega f^+ d\mu < +\infty$ or $\int_\Omega f^- d\mu < +\infty$. Furthermore, show that in this case, $\int_\Omega f d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu$.

Exercise 10.2. ♣
Which of the following statements are true?
(a) Let $f \in L^1([0,1]) \cap C^1([0,1])$ such that $\lim_{x\to 1^-} f(x) = a \in \mathbb{R}$. Then

$$\lim_{n\to\infty} n \int_0^1 x^n f(x) dx = a^2.$$

(b) Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}$ converging uniformly to a function f. Then $\lim_{k\to\infty} \int_\mathbb{R} f_k(x)dx$ exists and

$$\int_\mathbb{R} f(x)dx \le \lim_{k\to\infty} \int_\mathbb{R} f_k(x)dx.$$

(c) Let $f_k : [0,1] \to [0,1]$ be measurable functions for $k = 1, 2, \dots$ and suppose that $f_k \to f$ almost everywhere. Then $\lim_{k\to\infty} \int_{[0,1]} f_k(x)dx$ exists and

$$\int_{[0,1]} f(x)dx \le \lim_{k\to\infty} \int_{[0,1]} f_k(x)dx$$

(d) Let f be Lebesgue-summable on $\mathbb{R}$ and $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ be measurable subsets of $\mathbb{R}$. Then the limit $\lim_{n\to\infty} \int_{E_n} f(x)dx$ exists.

(e) Let $\{f_n\}$ be a sequence of continuous Lebesgue-summable functions on $[0,\infty)$ which converges uniformly to a Lebesgue-summable function f. Then

$$\lim_{n\to\infty} \int_{[0,\infty)} |f_n(x) - f(x)| dx = 0.$$

Exercise 10.3.
Let $f \in L^1(0,1)$. Compute

$$\lim_{k\to\infty} \int_0^1 k \log\left(1 + \frac{|f(x)|^2}{k^2}\right) dx.$$

Hint: You might want to use the following elementary inequality:

$$\log(1+t) \le 2\sqrt{t} \iff 1 + t \le e^{2\sqrt{t}} = 1 + 2\sqrt{t} + 2t + ..., \quad t \ge 0.$$

Exercise 10.4.
Let $f : \mathbb{R} \to [0, +\infty]$ be $\mathcal{L}^1$-measruable. Assume that for all $n \geq 1$,

$$\int_{\mathbb{R}} \frac{n^2}{n^2 + x^2} |f(x)| \, d\mathcal{L}^1(x) \leq 1.$$

Show that

$$\int_{\mathbb{R}} |f| \, d\mathcal{L}^1 \leq 1.$$

Exercise 10.5.
Compute the limit

$$\lim_{n\to\infty} \int_{[0,n]} \left(1 + \frac{x}{n}\right)^n e^{-2x} \, dx.$$

Exercise 10.6. ★
Let f_k, f be $\mathcal{L}^1$-summable functions on $\mathbb{R}$ which are nonnegative $\mathcal{L}^1$-almost everywhere and satisfy the following additional hypotheses:

- $\liminf_{k\to\infty} f_k(x) \geq f(x)$ for $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$.
- $\limsup_{k\to\infty} \int_{\mathbb{R}} f_k(x) \, dx \leq \int_{\mathbb{R}} f(x) \, dx$.

Show that

$$\lim_{k\to\infty} \int_{\mathbb{R}} |f_k(x) - f(x)| \, dx = 0.$$

Exercise 10.7. ★
Let $0 < m < M < \infty$ be two real numbers and let $f : [0, 1] \to \mathbb{R}$ be a measurable function satisfying $m \leq f(x) \leq M$ for almost every $x \in [0, 1]$. Show that

$$\left(\int_{[0,1]} f(x) \, dx\right) \left(\int_{[0,1]} \frac{1}{f(x)} \, dx\right) \leq \frac{(m + M)^2}{4mM}$$

and characterize all functions for which equality holds.

Exercise 10.8.
For all $n \in \mathbb{N}$, let $f_n \colon [0,1] \to \mathbb{R}$ be defined by:

$$f_n(x) = \frac{n\sqrt{x}}{1+n^2x^2}.$$

Prove that:
(a) $f_n(x) \leq \frac{1}{\sqrt{x}}$ on $(0,1]$ for all $n \geq 1$.

(b) $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = 0$.