Exercise 11.1. ♣

Let $\{f_k\}$ be a sequence of $\mathcal{L}^1$-measurable functions on $[0,\pi]$ converging a.e. to a function f.

(a) If $|f_k| \leq 100$ a.e. for each k, is it true that

$$\lim_{k\to\infty} \int_0^\pi |f_k - f| \, dx = 0?$$

(b) Is is true that

$$\lim_{n\to\infty} \int_0^\pi f_n(x) e^{-f_n(x)} dx = \int_0^\pi f(x) e^{-f(x)} dx?$$

(c) There exists no sequence of functions $(g_n)_{n\in\mathbb{N}} \subseteq L^1(\mathbb{R})$ that converges uniformly to zero on every compact set but $\int_{\mathbb{R}} g_n(x) dx = 1$ for all $n \in \mathbb{N}$.

(d) Assume that for all $n \geq 1$

$$\int_0^\pi \frac{n^2}{n + xe^{f(x)}} \ln\left(1 + \frac{|f(x)|^2}{n}\right) dx \leq 2.$$

What is the correct answer?

(A) $\|f\|_1 \leq 2$. (B) $\|f\|_2 \leq 2$. (C) $\|f\|_1 \leq \sqrt{2\pi}$. (D) None of the previous is correct.

(e) What is the value of the limit

$$\lim_{k\to\infty} \int_0^\infty e^{-x^k} \, dx?$$

(A) 0. (B) 1. (C) ∞. (D) None of the previous is correct.

(f) What is the value of the limit

$$\lim_{k\to\infty} k \int_{-\infty}^\infty \frac{e^{kx}}{1 + e^{2kx}} \exp\left(-\frac{|1-\cos(x)|}{x^2}\right) dx?$$

(A) 0. (B) $\frac{\pi}{2}e$. (C) ∞. (D) None of the previous is correct.

Exercise 11.2.
Compute the limit

$$\lim_{n\to\infty}\int_a^{+\infty}\frac{n}{1+n^2x^2}\,dx$$

for every $a\in\mathbb{R}$.
Hint: recall that $\arctan x$ is a primitive of $\frac{1}{1+x^2}$.

Exercise 11.3.
Let μ be a Radon measure on $\mathbb{R}^n$, $\Omega\subset\mathbb{R}^n$ be μ-measurable and $f\colon\Omega\to[0,+\infty]$ be μ-summable. For all μ-measurable subsets $A\subset\Omega$ define (see Section 3.5 in the Lecture Notes)

$$\nu(A)=\int_A f\,d\mu.$$

(a) Prove that ν is a pre-measure on the σ-algebra of μ-measurable sets, hence we can define its Carathéodory-Hahn extension $\nu\colon\mathcal{P}(\Omega)\to[0,+\infty]$.

(b) Show that ν is a Radon measure.

(c) Prove that $\Sigma_\nu\supseteq\Sigma_\mu$ and that ν is absolutely continuous with respect to μ, that is, if $\mu(A)=0$ then $\nu(A)=0$.

Exercise 11.4.
Prove the following assertions.
(a) Let $f\colon[a,+\infty)\to\mathbb{R}$ be a locally bounded function and locally Riemann integrable. Then f is $\mathcal{L}^1$-summable if and only if f is absolutely Riemann integrable in the generalized sense (namely $\mathcal{R}\int_a^\infty|f(x)|dx=\lim_{j\to\infty}\mathcal{R}\int_a^j|f(x)|dx$ exists and it is finite) and in this case

$$\int_{[a,+\infty)}f(x)d\mathcal{L}^1=\mathcal{R}\int_a^\infty f(x)dx=\lim_{j\to+\infty}\mathcal{R}\int_a^j f(x)dx.$$

(b) Let $f\colon[0,+\infty)\to\mathbb{R}$ be the function $f(x)=\frac{\sin x}{x}$, which is locally bounded and locally Riemann integrable. Show that f is Riemann integrable, i.e. $\mathcal{R}\int_0^\infty f(x)dx<+\infty$ but not absolutely Riemann integrable, i.e. $\mathcal{R}\int_0^\infty|f(x)|dx=\infty$. Hence f is not $\mathcal{L}^1$-summable.

Exercise 11.5.
This exercise is a more general version of Theorem 3.4.1 from the lecture notes.
(a) Let μ be a Radon measure on $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be a μ-measurable subset. Consider a function $f\colon \Omega \times (a,b) \to \mathbb{R}$, for some interval $(a,b) \subset \mathbb{R}$, such that:

- the map $x \mapsto f(x,y)$ is μ-summable for all $y \in (a,b)$;
- the map $y \mapsto f(x,y)$ is differentiable in (a,b) for every $x \in \Omega$;
- there is a μ-summable function $g\colon \Omega \to [0,\infty]$ such that $\sup_{a<y<b}|\frac{\partial f}{\partial y}(x,y)| \leq g(x)$ for all $x \in \Omega$.

Then $y \mapsto \int_\Omega f(x,y)d\mu(x)$ is differentiable in (a,b) with

$$\frac{d}{dy}\left(\int_\Omega f(x,y)d\mu(x)\right) = \int_\Omega \frac{\partial f}{\partial y}(x,y)d\mu(x)$$

for all $y \in (a,b)$.

(b) ★ Compute the integral

$$\phi(y) := \int_{(0,\infty)} e^{-x^2-y^2/x^2} d\mathcal{L}^1(x)$$

for all $y > 0$.

Hint: use part (a) to obtain that ϕ solves the Cauchy problem

$$\begin{cases} \phi'(y) = -2\phi(y) & \text{for } y > 0 \\ \lim_{y\to 0^+} \phi(y) = \sqrt{\pi}/2. \end{cases}$$

Exercise 11.6.
Let μ be a Radon measure on $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ a μ-measurable set with $\mu(\Omega) < +\infty$ and $f, f_k : \Omega \to \overline{\mathbb{R}}$ μ-summable functions.
(a) Show that Vitali's Theorem implies Dominated Convergence Theorem.

(b) Let $\Omega = [0,1]$ and $\mu = \mathcal{L}^1$. Give an example in which Vitali's Theorem can be applied but Dominated Convergence Theorem cannot, i.e., a dominating function does not exist.

Hint: look at the functions $f_n^k(x) = \frac{1}{x}\chi_{[\frac{n+k-1}{n2^{n+1}}, \frac{n+k}{n2^{n+1}})}(x)$ for $n \in \mathbb{N}$, $1 \leq k \leq n$.