Exercise 12.1. ♣

(a) Is the following equality true?

$$\lim_{n\to\infty}\int_0^1 e^{\frac{x^2}{n}}\,dx = \int_0^1 \lim_{n\to\infty} e^{\frac{x^2}{n}}\,dx.$$

(b) The value of the limit

$$\lim_{n\to\infty}\int_1^\infty \frac{\ln(nx)}{x+x^2\ln n}dx$$

is

(A) 0. (B) 1. (C) e. (D) $+\infty$.

(c) The value of the limit

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-\pi x}\,dx$$

is

(A) 0. (B) $\frac{1}{\pi-1}$. (C) $\frac{2}{\pi-1}$. (D) 1.

(d) The value of the limit

$$\lim_{n\to\infty}\int_0^\infty \left(\frac{\sin x}{x}\right)^n dx$$

is

(A) 0. (B) 1. (C) 2. (D) $+\infty$.

(e) For $p, q \geq 1$, let $(f_n)_{n\in\mathbb{N}} \subseteq L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ such that $f_n \to f$ in $L^p(\mathbb{R})$ and $f_n \to g$ in $L^q(\mathbb{R})$ and $f_n(x) \to h(x)$ for almost every $x \in \mathbb{R}$. Consider the following statements:

(i) $f(x) = g(x)$, for almost every $x \in \mathbb{R}$.
(ii) $f(x) = h(x)$, for almost every $x \in \mathbb{R}$.

Which of them are true?

(A) Both (i) and (ii).
(B) (i) but not (ii).
(C) (ii) but not (i).
(D) Neither (i) nor (ii).

(f)Consider the following statements:

(i) If $f \in L^p([0,1])$ for all $p \in (1,\infty)$, then $f \in L^\infty([0,1])$.

(ii) If $1 \leq p < q < +\infty$, then $L^q([1,\infty)) \subseteq L^p([1,\infty))$.

Which of them are true?

(A) Both (i) and (ii).

(B) (i) but not (ii).

(C) (ii) but not (i).

(D) Neither (i) nor (ii).

Exercise 12.2.
Evaluate

$$\sum_{n=0}^{\infty} \int_0^{\frac{\pi}{2}} \left(1 - \sqrt{\sin x}\right)^n \cos x \, dx.$$

Exercise 12.3.
Let $1 \leq p < \infty$. Show that if $\varphi \in L^p(\mathbb{R}^n)$ and φ is uniformly continuous, then

$$\lim_{|x| \to \infty} \varphi(x) = 0.$$

Exercise 12.4.
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ a μ-measurable set.
(a) (Generalized Hölder inequality) Consider $1 \leq p_1, \ldots, p_k \leq \infty$ such that $\frac{1}{r} = \sum_{i=1}^k \frac{1}{p_i} \leq 1$. Show that, given functions $f_i \in L^{p_i}(\Omega,\mu)$ for $i = 1, \ldots, k$, it holds $\prod_{i=1}^k f_i \in L^r(\Omega,\mu)$ and

$$\left\| \prod_{i=1}^k f_i \right\|_{L^r} \leq \prod_{i=1}^k \|f_i\|_{L^{p_i}}.$$

(b) Prove that, if $\mu(\Omega) < +\infty$, then $L^s(\Omega,\mu) \subseteq L^r(\Omega,\mu)$ for all $1 \leq r < s \leq +\infty$.

(c) Show that the inclusion in part (b) is strict for all $1 \leq r < s \leq +\infty$.

Exercise 12.5. ★
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ a μ-measurable set with $\mu(\Omega) < +\infty$. Consider a function $f \colon \Omega \to \overline{\mathbb{R}}$ such that $fg \in L^1(\Omega,\mu)$ for all $g \in L^p(\Omega,\mu)$. Prove that $f \in L^q(\Omega,\mu)$ for all $q \in [1,p')$, where $p' = \frac{p}{p-1}$ is the conjugate of p.

Exercise 12.6.
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ a μ-measurable set.
(a) Show that any $f \in \bigcap_{p\in\mathbb{N}^*} L^p(\Omega,\mu)$ with $\sup_{p\in\mathbb{N}^*} \|f\|_{L^p} < +\infty$ lies in $L^\infty(\Omega,\mu)$.

Hint: Tchebychev's inequality.

(b) ★ Show that if $\mu(\Omega) < +\infty$, then for any f as in part (a) we have that $\|f\|_{L^\infty} = \lim\limits_{p\to\infty} \|f\|_{L^p}$.

Exercise 12.7.
Let $(x_{n,m})_{(n,m)\in\mathbb{N}^2} \subset [0,+\infty]$ be a sequence parametrized by $\mathbb{N}^2$. Show that

$$\sum_{(n,m)\in\mathbb{N}^2} x_{n,m} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{n,m} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x_{n,m}.$$

Remark. Given a sequence $(x_\alpha)_{\alpha\in A} \subset [0,+\infty]$ parametrized by an arbitrary set A, we define

$$\sum_{\alpha\in A} x_\alpha := \sup_{F\,\subset\,A\text{ finite}} \sum_{\alpha\in F} x_\alpha.$$