Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

1.1. MC Questions

(a) Let $f(z) = z^2$. How does f map the upper half-plane (the set of complex numbers with positive imaginary parts)?

A) It maps the upper half-plane to the left half-plane.

B) It maps the upper half-plane to the lower half-plane.

C) It maps the upper half-plane to both the upper and lower half-planes.

D) It maps the upper half-plane to the entire complex plane except the real axis.

(b) Let $f(z) = u + iv$ be a holomorphic function in the unit disc $D_1(0)$. Let $F_1(z) := \overline{f(\bar{z})}$ and $F_2(z) := f(\bar{z})$. Which one of the following statemnets is correct.

A) F_1 is holomorphic but F_2 is not.

B) F_2 is holomorphic but F_1 is not.

C) Both F_1 and F_2 are holomorphic.

D) Neither F_1 nor F_2 is holomorphic.

1.2. Complex Numbers Review

(a) Simplify the following expressions

$$\left(1 + i\sqrt{3}\right)^{50} =$$

$$(1+i)^{2n}(1-i)^{2m} = \qquad \text{for every } m, n \in \mathbb{N}.$$

(b) Express the complex number $z = -1 + i\sqrt{3}$ in polar form and compute all of its cubic roots.

(c) Find all $z \in \mathbb{C}$ such that $z^2 + (3+4i)z + (5+6i) = 0$.

1.3. Power Series Investigate the absolute convergence and radius of convergence of the following power series

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{3^n} z^n, \qquad \sum_{n=0}^{+\infty} \frac{n!}{(2n)!} z^n, \qquad \sum_{n=0}^{+\infty} n^2 z^n.$$

1.4. Differentiability, Cauchy-Riemann and Holomorphicity Provide, with proof:

(a) some function $f\colon \mathbb{C} \to \mathbb{C}$ such that, for some $z_0 \in \mathbb{C}$, f satisfies the Cauchy-Riemann equations at z_0, but is *not* holomoprhic at z_0;

(b) functions $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ such that, for some $p_0 = (x_0, y_0) \in \mathbb{R}^2$, the function $(u, v)\colon (x, y) \mapsto (u(x, y), v(x, y))$ is differentiable at p_0, but the function $u+iv\colon \mathbb{C} \to \mathbb{C}$ defined for $z = x + iy$ by $(u + iv)(z) = (u + iv)(x + iy) = u(x, y) + i \cdot v(x, y)$ is *not* holomorphic at $z_0 = x_0 + iy_0$.

1.5. Applications of CR equations Let $\Omega \subset \mathbb{C}$ be a domain, i.e an open connected subset of $\mathbb{C}$.

(a) Let $u : \Omega \to \mathbb{R}$ be a differentiable function such that $\frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0$ for all $z \in \Omega$. Prove that u is constant on Ω.

(b) Let $f : \Omega \to \mathbb{C}$ be holomorphic and $f'(z) = 0$ for all $z \in \Omega$. Prove that f is constant in Ω.

(c) If $f = u+iv$ is holomorphic on Ω and if any of the functions u, v or $|f|$ is constant on Ω then f is constant.

1.6. ⋆ Geometric transformations of the complex plane

(a) Describe the transformation $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \frac{1}{z}$ in terms of geometric operations on the complex plane. Find the image of the unit circle under this transformation.

(b) Let $f\colon \mathbb{C} \setminus \{-1\} \to \mathbb{C}$ be defined as $f(z) = \frac{z-i}{z+i}$. Show that this transformation maps the upper half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ to the unit disk $\{z \in \mathbb{C} : |w| < 1\}$.

1.7. ⋆ Let $u : \mathbb{C} \to \mathbb{R}$ be a real-valued function on $\mathbb{C}$.

(a) Show that there is at most one holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $\mathrm{Re}(f) = u$ and $\mathrm{Im}(f(0)) = 0$.

(b) Give an example of a C^∞ function u such that there is no f as in the previous item.