Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

2.1. MC Questions

(a) Which of the following functions is NOT holomorphic?

A) $f(z) = z^6 + 5$

B) $f(z) = x^2 - y^2 + x + i(y + 2xy)$

C) $f(z) = (\cos(x) + i\sin(x))e^{-y}$

D) $f(z) = x - iy + 2$

(b) Given that the derivative of a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ is expressed as a power series

$$f'(z) = \sum_{n=0}^{\infty} b_n z^n,$$

which of the following is, for some value of the constant $C \in \mathbb{C}$, the correct expression for $f(z)$?

A) $f(z) = \sum_{n=0}^{\infty} \frac{b_n}{n+1} z^n + C$

B) $f(z) = \sum_{n=0}^{\infty} \frac{b_n}{n} z^{n+1} + C$

C) $f(z) = \sum_{n=0}^{\infty} b_n z^{n+1} + C$

D) $f(z) = \sum_{n=0}^{\infty} \frac{b_n}{n+1} z^{n+1} + C$

2.2. Complex numbers and geometry I Denote with $A_y := \{iy : y \in \mathbb{R}\} \subset \mathbb{C}$ the y-axis in the complex plane. Describe geometrically the image of A_y under the exponential map $\{e^z : z \in A_y\}$. Repeat the same replacing A_y with the x-axis $A_x := \{x : x \in \mathbb{R}\} \subset \mathbb{C}$, the diagonal $D := \{a + ia : a \in \mathbb{R}\} \subset \mathbb{C}$, and the curve $\{\log(a) + ia : a > 0\} \subset \mathbb{C}$.

2.3. Integrating over a triangle Let Ω be an open subset of $\mathbb{C}$. Suppose that $f : \Omega \to \mathbb{C}$ is holomorphic, and that $f' : \Omega \to \mathbb{C}$ is continuous. Show taking advantage of the Green formula [1] that

$$\int_T f\, dz = 0,$$

[1] Let C be a positively oriented, piecewise-smooth simple curve in the plane, and let D be the region bounded by C. If $\vec{F} = (F^1, F^2) : \bar{D} \to \mathbb{R}^2$ is a vector field whose components have continuous partial derivatives, then Green's theorem states: $\int_C \vec{F} \cdot dr = \iint_D (\partial_x F^2 - \partial_y F^1)\, dx\, dy$.

where the integration is along an arbitrary triangle T contained in Ω.

2.4. * Line integral I Compute the following complex line integrals. Here $\Re(z)$ and $\Im(z)$ denote respectively the real and imaginary parts of z.

(a) $\int_\gamma (z^2+z)\,dz$, when γ is the segment joining 1 to $1+i$.

(b) $\int_\gamma \bar{z}\,dz$, when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

(c) $\int_\gamma z^n\,dz$, when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and $n \in \mathbb{Z}$.

(d) $\int_\gamma z^n\,dz$, when γ is the circle $\{z \in \mathbb{C} : |z-2| = 1\}$ and $n \in \mathbb{N}$.

(e) $\int_\gamma \frac{dz}{(z-a)(z-b)}$ when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and $a, b \in \mathbb{C}$ with $|a| < 1 < |b|$.

2.5. Line Integral II Is it true that for any $f : \mathbb{C} \mapsto \mathbb{C}$

$$\Re \int_\gamma f(z)\,dz = \int_\gamma \Re(f(z)\,dz$$

If so prove it, if not give a counterexample.

2.6. * Differentiability

(a) Prove (without using the Cauchy-Riemann equation) that the functions

$$f(z) = \Re(z), \quad g(z) = \Im(z)$$

are not differentiable at any point.

(b) Let $a, b \in \mathbb{C}$. Find all points in $\mathbb{C}$ where $af(z) + bg(z)$ is differentiable.