Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

3.1. MC Questions

(a) For which values of $z \in \mathbb{C}$ is $\cos(z)$ a real number?

A) Only for $z = x$ with $x \in \mathbb{R}$.

B) Only for $z = x + iy$ with $y = 0$ and $x = n\pi$ for some $n \in \mathbb{Z}$.

C) Only for $z = iy$ with $y \in \mathbb{R}$.

D) Only for $z = x + iy$ with $y = 0$ or $x = n\pi$ for some $n \in \mathbb{Z}$.

(b) Let $a, b \in \mathbb{C}$ with $a \neq b$. Define $\nu = \frac{a-b}{|a-b|}$ and let γ be a parametrization of the line segment from a to b. Moreover, let $f: \mathbb{C} \to \mathbb{C}$ be a continuous function. Which of the following equalities holds?

A) $\int_\gamma f(z)dz = \int_0^{|b-a|} f(a + t\nu)dt$

B) $\int_\gamma f(z)dz = (b-a) \int \int_0^1 f(a(1-t) + bt)dt$

C) $\int_\gamma f(z)dz = \int_0^1 f(a(1-t) + bt)dt$

D) $\int_\gamma f(z)dz = \int_{|a|}^{|b|} f(t\nu)dt$

3.2. Complex line integrals

(a) Compute $\int_\gamma \cos(\Im(z))\, dz$, when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

(b) Compute $\int_\gamma (\bar{z})^k\, dz$ for any $k \in \mathbb{Z}$ and when γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

(c) Compute $\int_\gamma (z^{2024} + \pi z^{13} + 1)\, dz$, when γ is the spiral $\{1 + te^{i\pi t} : t \in [0, 1]\}$.

3.3. Show that $exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. What's the image of $\cos\colon \mathbb{C} \to \mathbb{C}$?

3.4. Let $MNPQ$ be a rectangle on the complex plane whose sides are parallel to the x-axis and y-axis. It is divided into smaller rectangles whose sides are parallel to the axes as well. It is known that each smaller rectangle has at least one side (horizontal or vertical) whose length belongs to the integers. Prove that $MNPQ$ also has at least one side of integer length.

Hint: $\int_a^b e^{2\pi i x} dx = 0 \iff b - a \in \mathbb{Z}$

3.5. $\star$ Harmonicity

(a) A real C^2-function $w = w(x,y) : \mathbb{R}^2 \to \mathbb{R}$ is said to be *harmonic* if its Laplacian $\Delta w = \operatorname{div}(\nabla w) := \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ is equal to zero everywhere. Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic function. Denote with $u = \Re(f)$ and $v = \Im(f)$ the real part and imaginary part of f, so that $f(z) = u(z) + iv(z)$ for every $z \in \mathbb{C}$. Show that both u and v are harmonic functions by identifying $\mathbb{C}$ with $\mathbb{R}^2$.

(b) Let D be the unit disk centered at the origin and let $h \colon D \to \mathbb{R}$ be a C^∞ harmonic function. Show that there exists some holomorphic function $F \colon D \to \mathbb{C}$ such that $h = \Re(F)$.

3.6. ⋆ Real integrals via complex integration

(a) (i) Show that

$$\int_0^\infty \frac{\sin(x)}{x} dx = \lim_{R\to+\infty} \frac{1}{2i} \int_{-R}^{R} \frac{e^{ix}-1}{x} dx.$$

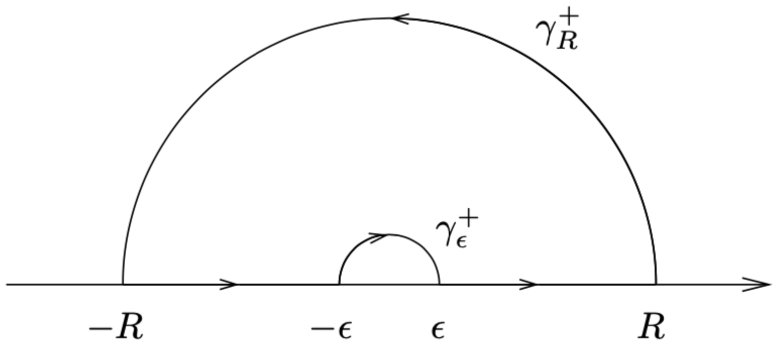
(ii) Let $R > 0$ be large and $\varepsilon > 0$ be small. Explain why

$$\int_\gamma \frac{e^{iz}-1}{z} dz = 0,$$

where γ is the "indented semicircle" curve described in the picture below.

(iii) Deduce the value of

$$\int_0^\infty \frac{\sin(x)}{x} dx.$$



(b) Let γ be the counter clockwise oriented unit circle and $n \in \mathbb{N}$. Compute

$$\int_\gamma z^{-1}(z - z^{-1})^n \, dz,$$

and deduce that

$$\int_0^{2\pi} \sin(t)^n \, dt = \begin{cases} \frac{\pi}{2^{n-1}}\binom{n}{n/2}, & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$