Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

5.1. MC Questions

(a) A subset $\mathcal{A}$ of a domain $\Omega \subset \mathbb{C}$ is called *discrete* in Ω if it has no limit point in Ω. For how many of the following pairs $(\Omega, \mathcal{A})$ is it not true that $\mathcal{A}$ is discrete in Ω?

(i) $\Omega = \mathbb{C}$. Define $q_n = \sum_{k=1}^{n} \frac{1}{k}$ and $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \{q_n e^{ix} : x = \frac{k2\pi}{n} \text{ for some } k \in \mathbb{N}\}$.

(ii) $\Omega = \mathbb{C}$. Define $q_n = \sum_{k=1}^{n} \frac{1}{k^2}$ and $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \{q_n e^{ix} : x = \frac{k2\pi}{n} \text{ for some } k \in \mathbb{N}\}$.

(iii) $\Omega = \mathbb{C}$, $\mathcal{A} = \{\frac{1}{n} : n \in \mathbb{N}^+\}$.

(iv) $\Omega = \mathbb{C}\backslash\{0\}$, $\mathcal{A} = \{\frac{1}{n} : n \in \mathbb{N}^+\}$.

A) 0

B) 1

C) 2

D) 3

(b) Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function on an open set Ω . For $w \in \mathbb{C}$ we define the set

$$E_w := \{z \in \Omega : f(z) = w.\}$$

Which of the following is true?

A) E_w is a discrete set in Ω only for $w = 0$.

B) E_w is a discrete set in Ω only for $w \neq 0$.

C) E_w is a discrete set in Ω for every $w \in \mathbb{C}$.

D) If Ω is connected then E_w is a discrete set in Ω for every $w \in \mathbb{C}$.

5.2. Order of zeros

(a) Find the zeros of the function $z \mapsto \sin(z^2)$ and determine their order.

(b) Let $p(z) := 1 + a_1 z + \cdots + a_n z^n$ be a polynomial and $f(z) := e^z - p(z)$. Clearly $z_0 = 0$ is a zero of the function $f(z)$. Compute $\mathrm{ord}_{z_0} f$, the order of the zero of f at z_0, as a function of the coefficients of $p(z)$.

5.3. ⋆ **The complex logarithm** Let

$$U = \mathbb{C} \setminus \{z \in \mathbb{C} : \Im(z) = 0, \Re(z) \leq 0\}$$

be the open set obtained by removing the negative real axis from the complex plane $\mathbb{C}$. The complex logarithm is defined in U as

$$\log(z) := \log(|z|) + i\arg(z), \quad z = |z|e^{i\arg(z)},$$

where $\arg(z) \in]-\pi, \pi[$. Show that for every $z \in U$

$$\log(z) = \int_\gamma \frac{1}{w}\, dw,$$

where γ is the segment connecting 1 to z.

Hint: integrate over a well chosen closed curve containing γ and passing through $|z|$.

5.4. A complex ODE Take advantage of the power series expansion around zero to find a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $f'(z) = zf(z)$ and $f(0) = 1$.

5.5. Riemann continuation Theorem Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be holomorphic. Show that the following are equivalent:

1. There exists $g : \mathbb{C} \to \mathbb{C}$ holomorphic, such that $g(z) = f(z)$ for all $z \neq 0$.
2. There exists $g : \mathbb{C} \to \mathbb{C}$ continuous, such that $g(z) = f(z)$ for all $z \neq 0$.
3. There exists $\varepsilon > 0$ such that f is bounded in $\dot{B}_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\} \setminus \{0\}$.
4. $\lim_{z \to 0} zf(z) = 0$.

Hint: to prove 4. ⇒ 1. define $h(z) = zf(z)$ when $z \neq 0$ and $h(0) = 0$. Analyse the relation between $f(z)$, $h(z)$ and $k(z) := zh(z)$.

5.6. ⋆ Let $D \subset \mathbb{C}$ be the unit disk at the origin. Find all functions $f(z)$ which are holomorphic on D and which satisfy

$$f\left(\frac{1}{n}\right) = n^2 f\left(\frac{1}{n}\right)^3, \quad n = 2, 3, 4, \ldots$$