Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

6.1. MC Questions

(a) Consider the sequence of functions $f_n(z) = \frac{z^n}{n+1}$ on $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Which of the following is true?

A) The sequence $\{f_n(z)\}$ converges uniformly on D.

B) The sequence $\{f_n(z)\}$ converges locally uniformly on D, but not uniformly.

C) The sequence $\{f_n(z)\}$ converges pointwise but not uniformly on D.

D) The sequence $\{f_n(z)\}$ does not converge on D.

(b) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n\colon \mathbb{C} \to \mathbb{C}$. Which of the following statements is true?

A) If $\sum_{n\in\mathbb{N}} f_n$ converges uniformly on $\mathbb{C}$ to some f and for each $n \in \mathbb{N}$ there exists some $M_n \in \mathbb{R}^+$ such that $\sup_{z\in\mathbb{C}} |f_n(z)| \geq M_n$, then $\sum_{n\in\mathbb{N}} M_n$ converges.

B) If $\sum_{n\in\mathbb{N}} f_n$ converges locally uniformly on $\mathbb{C}$ to some f and for each $n \in \mathbb{N}$ there exists some $M_n \in \mathbb{R}^+$ such that $\sup_{z\in\mathbb{C}} |f_n(z)| \geq M_n$, then $\sup_{n\in\mathbb{N}} M_n$ is finite, i.e. the sequence $(M_n)_{n\in\mathbb{N}}$ is bounded.

C) If for all $n \in \mathbb{N}$ we have that $\sup_{z\in\mathbb{C}} |f_n(z)| = +\infty$, then $\sum_{n\in\mathbb{N}} f_n$ can not converge uniformly to any $f\colon \mathbb{C} \to \mathbb{C}$.

D) If for all $n \in \mathbb{N}$ we have that $\sup_{z\in\mathbb{C}} |f_n(z)| = +\infty$, then $\sum_{n\in\mathbb{N}} f_n$ converges uniformly to some $f\colon \mathbb{C} \to \mathbb{C}$.

6.2. Show that the following functions exist and are holomorphic on the indicated open sets; furthermore, give a similar expression for their derivatives:

(c) $f_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$ on $D_1(0)$

$\star$ **(b)** $f_2(z) = \int_0^1 (1-tz)^4 e^{tz}\, dt$ on $\mathbb{C}$

$\star$ **(c)** $f_3(z) = \sum_{n=0}^{\infty} n^2 \exp(2i\pi n^3 z)$ on $H = \{z \in \mathbb{C} \mid \Im(z) > 0\}$

6.3.

(a) Prove that the sequence $f_n(z) = z^n$, $n \geq 1$ converges locally uniformly but not uniformly on $\{z : |z| < 1\}$.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary (not necessarily continuous) function and define for $n \in \mathbb{N}$

$$f_n(z) = \begin{cases} f(z), & \text{if } |z| \leq n, \\ 0, & \text{if } |z| > n. \end{cases}$$

Show that the sequence (f_n) converges pointwise and locally uniformly to f, and that it converges uniformly to f if and only if $\lim_{|z|\to\infty} f(z) = 0$.

6.4. $\star$

Let f be a holomorphic function on $D = \{z : |z| < 1\}$ with $f(0) = 0$. Prove that the series $\phi(z) = \sum_{n=1}^{\infty} f(z^n)$ converges locally uniformly on D.

6.5. Weierstrass M-test Let $f_n \colon A \to \mathbb{C}$ be a sequence of functions and M_n be a sequence of real numbers such that

$$|f_n(z)| \leq M_n, \ \forall n \geq 1, \ \forall z \in A \ \text{ and } \sum_{n=1}^{\infty} M_n \text{ converges.}$$

Prove that $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly on A.