Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

7.1. MC Questions

(a) Consider the function $f\colon z \mapsto \frac{e^z}{e^z-1}$, defined for all $z \in \mathbb{C}$ such that $e^z \neq 1$. Which of the following statements holds?

A) The function f is holomorphic on $\mathbb{C} \setminus \{0\}$.

B) The function f has poles and each pole is simple.

C) The function f has both poles and removable singularities.

D) The function f has finitely many singularities.

(b) Which of the following equalities is **false**?

A) $\operatorname{res}_{2i}\left(\frac{1}{z^2+4}\right) = \frac{1}{4i}$

B) $\operatorname{res}_{0}\left(\frac{\sin(z)}{z^2}\right) = 1$

C) $\operatorname{res}_{0}\left(\frac{\cos(z)}{z^2}\right) = 0$

D) $\operatorname{res}_{1}\left(\frac{1}{z^5-1}\right) = \frac{1}{5!}$

7.2. Schwarz reflection principle Let Ω be open, **connected**, and symmetric with respect to the x-axis (i.e. $z \mapsto \bar{z}$ preserves Ω), and let $f : \Omega \to \mathbb{C}$ be holomorphic. Let $L := \{z \in \Omega : \Im(z) = 0\}$. Note that L is non-empty. Prove that $f(\bar{z}) = \overline{f(z)}$ for all $z \in \Omega$ if and only if f is real valued on L.

Hint: consider g to be the restriction of f to the upper half plane intersected with Ω. 'Reflect' g by imposing $g^*(z) := \overline{g(\bar{z})}$. Argue taking advantage of Morera's Theorem.

7.3. $\star$ **Dense image** Show that the image of a non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is *dense* in $\mathbb{C}$, that is: for every $z \in \mathbb{C}$ and $\varepsilon > 0$, there exists $w \in \mathbb{C}$ such that $|z - f(w)| < \varepsilon$.

Remark: In fact there is a theorem, called the little Picard Theorem, which asserts that $f(\mathbb{C})$ misses at most one single point of $\mathbb{C}$!

7.4. Let $D = \{z \mid |z| < 1\}$ be the unit disk, and let $\overline{D}$ be its closure. Give an example of a continuous function $f : \overline{D} \to \mathbb{C}$ that is holomorphic on D, but does not have a holomorphic continuation on any domain in $\mathbb{C}$ containing $\overline{D}$.

7.5. $\star$ **Complex integrals** Compute the following complex integrals taking advantage of the Residue Theorem.

(a)

$$\int_{|z|=2} \frac{e^z}{z^2(z-1)}\, dz.$$

(b)

$$\int_{|z|=1} \frac{1}{z^2(z^2-4)e^z}\, dz.$$

(c)

$$\int_{|z|=1/2} \frac{1}{z\sin(1/z)}\, dz.$$

Hint: Note that the function $\frac{1}{z\sin(1/z)}$ has infinitely many singularities accumulating at 0. Hence you cannot use the residue theorem directly. To go around this problem first prove

$$\int_{|z|=1/2} \frac{1}{z\sin(1/z)}\, dz = \int_{|w|=2} \frac{1}{w\sin(w)}\, dw.$$

(d)

$$\int_{|z|=5} \frac{1}{(z-i)(z+2)(z-4)}\, dz.$$

7.6. The Gamma function Let $Z_- := \{0, -1, -2, \dots\}$ the set of all non-positive integers, and define for all $\tau \in \mathbb{R}$ the set $U_\tau := \{z \in \mathbb{C} : \Re(z) > \tau, z \notin Z_-\}$, and $U := \mathbb{C} \setminus Z_-$.

(a) Show that the function defined by the complex improper Riemann integral

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1}\, dt$$

is well defined for all $z \in U_1$. (Here $t^{z-1} = \exp((z-1)\log(t))$).

(b) Prove that Γ is holomorphic in U_1.

Hint: First show that the functions of the sequence $(\Gamma_n)_{n\in\mathbb{N}}$ *given by truncating the integral at height* n *(*$\Gamma_n(z) = \int_0^n e^{-t}t^{z-1}\,dt$*) are holomorphic. Then, show that* $\Gamma_n \to \Gamma$ *uniformly in all compact subsets of* U_1*.*

(c) Show that $\Gamma(z+1) = z\Gamma(z)$ for all $z \in U_1$.

(d) Deduce that Γ allows a unique holomorphic extension to U_0.

(e) Deduce that Γ allows a unique holomorphic extension to U.