Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

## 8.1. MC Questions

**(a)** Consider the real integral $I = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$. This can be computed using Cauchy's Residue Theorem. Which of the following is false?

A) Let $\gamma(R)$ be a closed semicircle of radius $R > 1$ (centered at the origin) in the lower half of the complex plane, which is traced counterclockwise. Then $I = \lim_{R\to\infty} \oint_{\gamma(R)} \frac{1}{1+z^2}\,dz$.

B) Let $\gamma(R)$ be a closed semicircle of radius $R > 1$(centered at the origin) in the lower half of the complex plane, traced clockwise. Then $I = \lim_{R\to\infty} \oint_{\gamma(R)} \frac{1}{1+z^2}\,dz$.

C) $I = 2\pi i \operatorname{Res}\left(\frac{1}{1+z^2}, z = i\right)$

D) $I = -2\pi i \operatorname{Res}\left(\frac{1}{1+z^2}, z = -i\right)$

**(b)** Let $f(z) = \frac{e^z}{(z-1)^3}$. What is the order of the pole of $f(z)$ at $z = 1$?

A) 0 (no pole)

B) 1

C) 2

D) 3

**8.2. Poles at infinity** Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. We say that $f$ *has a pole at infinity of order* $N \in \mathbb{N}$ if the function $g(z) := f(1/z)$ has a pole of order $N$ at the origin in the usual sense. Prove that if $f : \mathbb{C} \to \mathbb{C}$ has a pole of order $N \in \mathbb{N}$ at infinity, then it has to be a polynomial of degree $N \in \mathbb{N}$.

**8.3. Meromorphic functions** For $z \in \mathbb{C}$ such that $\sin(z) \neq 0$ define the map

$$\operatorname{cotan}(z) = \frac{\cos(z)}{\sin(z)}.$$

**(a)** Show that cotan is meromorphic in $\mathbb{C}$, determine its poles and their residues.

**(b)** Let $w \in \mathbb{C} \setminus \mathbb{Z}$ and define

$$f(z) = \frac{\pi \operatorname{cotan}(\pi z)}{(z+w)^2}.$$

Show that $f$ is meromorphic in $\mathbb{C}$, determine its poles and their residues.

**(c)** Compute for every integer $n \geq 1$ such that $|w| < n$ the line integral

$$\int_{\gamma_n} f \, dz,$$

where $\gamma_n$ is the circle or radius $n + 1/2$ centered at the origin and positively oriented.

**(d)** Deduce that

$$\lim_{n \to +\infty} \sum_{k=-n}^{n} \frac{1}{(w+k)^2} = \frac{\pi^2}{\sin(\pi w)^2}.$$

**8.4.** ⋆ **Real integrals** Compute the following real integrals taking advantage of the Residue Theorem[1].

**(a)**

$$\int_0^{2\pi} \frac{1}{1+\sin^2(t)} \, dt$$

**(b)**

$$\int_{-\infty}^{\infty} \frac{e^{-x^2}}{x^2+1} \, dx$$

**8.5. Quotient of holomorphic functions** Let $f, g$ be two non-constant holomorphic functions on $\mathbb{C}$. Show that if $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$, then there exists $c \in \mathbb{C}$ such that $f(z) = cg(z)$.

**8.6.** ⋆ Let $P(z)$ be a complex polynomial of degree $n$ and $R > 0$ so large that $P(z)$ does not vanish in $\{z : |z| \geq R\}$. Let $\gamma$ be the path with $\gamma(t) = Re^{it}$ , with $0 \leq t \leq 2\pi$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{P'(z)}{P(z)} dz = n.$$

[1] Recall: $\{z_1, \dots, z_N\} \subset \Omega$ poles and $f : \Omega \setminus \{z_1, \dots, z_N\} \to \mathbb{C}$ holomorphic. Then if $\{z_1, \dots, z_N\}$ are inside a simple closed curve $\gamma$ in $\Omega$, then $\int_\gamma f \, dz = 2\pi i \sum_{j=1}^{N} \operatorname{res}_{z_j}(f)$.