Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

9.1. MC Questions

(a) Suppose $f(z) = z^3 + 3z + 2$ and $g(z) = z^3 + 2$. What's the number of zeros of $f(z) + g(z)$ inside $|z| = 2$?

A) 0

B) 6

C) 1

D) 3

(b) Let $f(z) : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function on $\mathbb{C}$. Which of the following conditions is both necessary **and** sufficient for $f(z)$ to be a rational function?

A) $f(z)$ has no singularities on $\mathbb{C}$.

B) $f(z)$ is holomorphic everywhere except for a finite number of poles.

C) $f(z)$ has at most a pole at infinity and at most finitely many poles in $\mathbb{C}$.

D) $f(z)$ has finitely many singularities.

9.2. Laurent Series A *Laurent series* centered at $z_0 \in \mathbb{C}$ is a series of the form

$$\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

where $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$. We define $\rho_0,\ \rho_I \in [0,+\infty]$ the *outer* and *inner* radius of convergence as

$$\rho_0 := \left(\limsup_{n\to+\infty} |a_n|^{1/n}\right)^{-1}, \qquad \rho_I := \limsup_{n\to+\infty} |a_{-n}|^{1/n}.$$

If $\rho_I < \rho_0$, we define the *annulus of convergence* as

$$\mathcal{A}(z_0,\rho_I,\rho_0) := \{z \in \mathbb{C} : \rho_I < |z-z_0| < \rho_0\},$$

with the convention $\mathcal{A}(z_0,\rho_I,+\infty) = \{z \in \mathbb{C} : \rho_I < |z-z_0|\}$, so that in particular $\mathcal{A}(z_0,0,+\infty) = \mathbb{C} \setminus \{z_0\}$.

(a) Show that if $\rho_0 > 0$, then the series

$$f_0(z) := \sum_{n=0}^{+\infty} a_n (z-z_0)^n, \quad z \in \mathcal{D}_0(z_0,\rho_0) := \{z \in \mathbb{C} : |z-z_0| < \rho_0\},$$

converges absolutely and uniformly on compact sets. Show that if $\rho_I < +\infty$, then the series

$$f_I(z) := \sum_{n=1}^{+\infty} a_{-n} (z-z_0)^{-n}, \quad z \in \mathcal{D}_I(z_0,\rho_I) := \{z \in \mathbb{C} : \rho_I < |z-z_0|\},$$

converges absolutely and uniformly on compact sets.

(b) Show that a Laurent series is divergent for any z satisfying $|z-z_0| > \rho_0$ or $|z-z_0| < \rho_I$.

(c) Deduce that the full Laurent series

$$f(z) := \sum_{n\in\mathbb{Z}} a_n (z-z_0)^n$$

defines an analytic function in $\mathcal{A}(z_0,\rho_I,\rho_0)$, and its coefficients are related to f by the formula

$$a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz,$$

for any $n \in \mathbb{Z}$ and $r \in (\rho_I, \rho_0)$.

9.3. Meromorphic functions Recall the definition of $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

(a) Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic. Show that f has at most countably many poles.

(b) Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be meromorphic on $\hat{\mathbb{C}}$. Show that f has at most finitely many poles.

(c) Deduce that if $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic on $\hat{\mathbb{C}}$, than it is a rational function.

9.4. ⋆ Generalization of the Argument Principle

(a) Let $\Omega \subset \mathbb{C}$ open, $z_0 \in \Omega$ and $r > 0$ such that $\bar{D}(z_0,r) = \{z \in \mathbb{C} : |z-z_0| \leq r\} \subset \Omega$. Suppose that $f : \Omega \to \mathbb{C}$ is homolorphic and that $f(z) \neq 0$ on the circle $\partial D(z_0,r) = \{z \in \mathbb{C} : |z-z_0| = r\}$. Show that for any holomorphic function $\varphi : \Omega \to \mathbb{C}$ we have that

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f'}{f}\varphi\,dz = \sum_{w\in D(z_0,r):f(w)=0} (\operatorname{ord}_w f)\varphi(w).$$

(b) Compute

$$\int_{|z|=2} \frac{ze^{z^3+1}}{z^2+1} dz$$

9.5. ⋆ Application of Rouché Theorem[1] Let $f(z)$ be a holomorphic function inside the unit disk $|z| < 1$, with the Taylor series expansion:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Suppose $f(z)$ is continuous on the closed unit disk and that it has exactly m zeros (counted with multiplicity) inside $|z| < 1$. Prove that:

$$\min_{|z|=1} |f(z)| \leq |c_0| + |c_1| + \cdots + |c_m|.$$

[1]Recall: Let $f, g : \Omega \to \mathbb{C}$ holomorphic and γ a closed, simple curve in Ω such that its interior lies in Ω. If $|f(z)| > |g(z)|$ for all $z \in \gamma$, then f and $f + g$ have the same number of zeros in the interior of γ.