Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

10.1. Multiple Choice

(a) Let f be a holomorphic function on the closed unit disk $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Consider the relationship between the location of the minimum modulus of $|f|$ and whether f has zeroes inside D. Which of the following statements is **true**?

A) If f has zeroes inside D, then the minimum modulus of $|f|$ is attained on the boundary ∂D.

B) If f has no zeroes inside D, then the minimum modulus of $|f|$ is attained on the boundary ∂D.

C) If the minimum modulus of $|f|$ is attained on the boundary ∂D, then f has zeroes inside D.

D) If the minimum modulus of $|f|$ is attained on the boundary ∂D, then f has no zeroes inside D.

(b) Which of the following sets is not simply connected?

A) $\{z = x + iy \in \mathbb{C} \mid 0 < y < x^2 \text{ or } x = 0\}$

B) $\mathbb{C} \setminus \{re^{i\theta} : r > 0, \theta = \pi/4\}$

C) $\{z = x + iy \in \mathbb{C} \mid |x| < 1, |y| < 1\}$

D) $\{z \in \mathbb{C} \mid |z| > 1\}$

10.2. Laurent Series II Let $0 \leq s_1 < r_1 < r_2 < s_2$, and set $U = \mathcal{A}(0, s_1, s_2)$ and $V = \mathcal{A}(0, r_1, r_2)$ (like in Exercise 9.1). Denote with γ_1 and γ_2 the circles of radius r_1 and r_2, respectively, positively oriented. Let $f : U \to \mathbb{C}$ be a general holomorphic function.

(a) Show that the functions

$$g_1(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w - z} \, dw, \quad \text{for } |z| > r_1,$$

and

$$g_2(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w - z} \, dw, \quad \text{for } |z| < r_2,$$

are well defined and holomorphic.

(b) Let γ be the closed curve obtained by going along γ_2 starting at r_2, then along the segment joining r_2 to r_1, then along $-\gamma_1$, and finally back via the segment joining r_1 to r_2. Let $z_0 \in V$ and $r > 0$ small enough such that $\sigma = \{z \in \mathbb{C} : |z - z_0| = r\}$ is in V. Explain why σ and γ are homotopic in U.

(c) Show that $f = g_2 - g_1$ in V.

(d) Deduce that f can be represented as a Laurent serie, meaning: there exists a sequence $(a_n)_{n\in\mathbb{Z}}$ such that the series $\sum_{n\geq 1} a_n z^n$ and $\sum_{n\geq 1} a_{-n} z^{-n}$ are absolutely convergent in V, and satisfy

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n, \qquad \text{in } V.$$

10.3. Complex vs Real Is it true that if $u, v : \mathbb{C} \to \mathbb{R}$ are smooth and open maps, then $f = u + iv$ is open? Answer from the perspective of the Open Mapping Theorem.

10.4. Maps preserving orthogonality Let $\Omega \in \mathbb{R}^2$ open, and $f : \Omega \to \mathbb{R}^2$ smooth. Show that if f is orientation preserving [1] and sends curves intersecting orthogonally to curves intersecting orthogonally, then f is holomorphic (by identifying $\mathbb{R}^2$ with $\mathbb{C}$).

10.5. $\star$ Let A be a square centered at the origin. Denote by s **one** arbitrarily fixed side of A. Let $f : A \to \mathbb{C}$ be holomorphic on the interior of A and continuous on the boundary of A, such that $f(z) = 0$ for all $z \in s$. Prove that $f = 0$ on A.

10.6. $\star$ **Multiple Ways** Let f be a holomorphic function on the closed unit disk $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. This exercise asks you to prove that

$$\max_{|z|=1} \left| f(z) - \frac{e^z}{z} \right| \geq 1$$

in **two** out of the three following ways:

(a) prove the claim using the Maximum Principle;

(b) prove the claim using Rouché's Theorem;

(c) prove the claim using Cauchy's Integral Formula.

[1] That is the determinant of its Jacobian is positive.