Exercises with a $\star$ are eligible for bonus points. Exactly one answer to each MC question is correct.

11.1. MC Questions

(a) Consider the principal branch of the complex-valued logarithm, with domain $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Let z, z_1, z_2 be elements of Ω, and let n be a positive natural number. Which of the following equalities does not always hold?

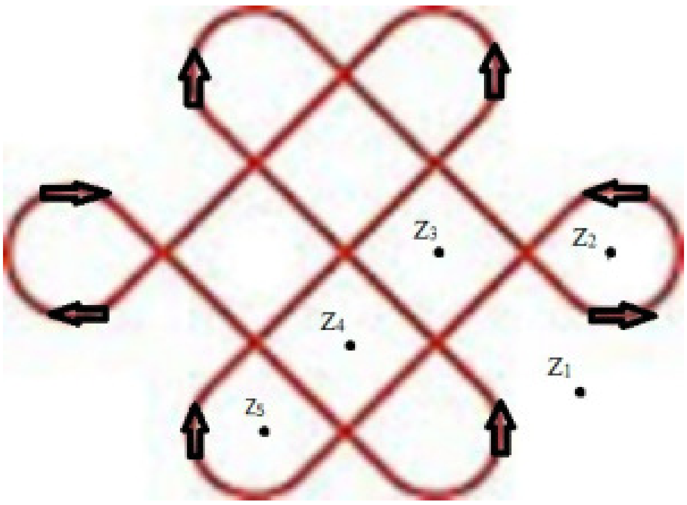
A) $e^{\log(z_1)+\log(z_2)} = z_1 z_2$

B) $\log(e^{\log(z_1)+\log(z_2)}) = \log(z_1) + \log(z_2)$

C) $(\log(z))' = \frac{1}{z}$

D) $\log(z/n) = \log(z) - \log(n)$

(b) Consider the following image, depicting five points $z_1, z_2, z_3, z_4, z_5 \in \mathbb{C}$ and a smooth path γ. What's the sum of the winding numbers $\sum_{i=1}^{5} W_\gamma(z_i)$?



A) -1

B) 0

C) 1

D) 2

11.2. Logarithm Let U be an open and simply connected domain of $\mathbb{C}$, and $f : U \to \mathbb{C}$ a non-vanishing holomorphic function. Fix $z_0 \in U$ and denote with γ_z an arbitrary curve in U connecting z_0 to z.

(a) Show that the function

$$g(z) = \int_{\gamma_z} \frac{f'}{f}\, dw,$$

is well defined and holomorphic in U, and that $g'(z) = \frac{f'(z)}{f(z)}$ for all $z \in U$.

(b) Compute the derivative of $\frac{\exp(g(z))}{f(z)}$.

(c) Deduce that there exists $\tilde{g}$ holomorphic in U such that $f = \exp(\tilde{g})$. Is this function unique?

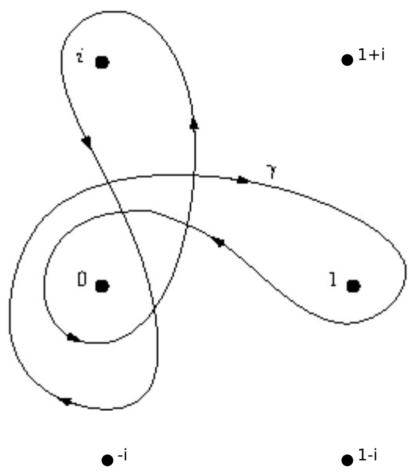
(d) Show that for every $n \in \mathbb{N}$ there exists an holomorphic function $h_n : U \to \mathbb{C}$ such that $(h_n)^n = f$.

11.3. $\star$ **Complex integral** Evaluate

$$\int_{|z|=1} \frac{z^8 - 3iz}{4\pi z^9 + 5z^5 - 4z^3 - 2i}\, dz.$$

Hint: take advantage of Rouché's Theorem and the Homotopy Theorem.

11.4. Winding number Evaluate the integral $\int_\gamma g\, dz$ when $g(z) = \frac{z}{(z^2+1)(z^2-2z+2)}$ and γ is as follows:



11.5. ⋆ **Fractional Residues** Prove the following: if z_0 is a simple pole of a meromorphic function f and A_ε is an arc of the circle $\{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$ of an angle $\alpha \in (0, 2\pi]$, then

$$\lim_{\varepsilon \to 0} \int_{A_\varepsilon} f \, dz = \alpha i \, \mathrm{res}_{z_0}(f).$$

11.6. Real integral Evaluate

$$\int_{-\infty}^{+\infty} \frac{\sin(x)}{x(x-\pi)} \, dx.$$

Hint: take a suitable contour in $\mathbb{C}$ that avoids the zeros of the denominator. Take advantage of Exercise 11.5.

11.7. Real integral II Let $\alpha \in (0, 1)$. Evaluate

$$\int_0^{+\infty} \frac{x^{2\alpha-1}}{1+x^2} \, dx,$$

choosing a suitable branch of the logarithm.