Exactly one answer to each MC question is correct.

12.1. MC Questions

(a) Suppose f is a conformal map from a simply connected region $\Omega \subset \mathbb{C}$ to the unit disc $\mathbb{D}$, and $f(z_0) = 0$. What condition determines the uniqueness of f?

A) f has a constant second derivative.

B) $f'(z_0) > 0$.

C) f extends to a continuous bijection on $\partial\Omega$.

D) f maps all boundary points of Ω to distinct points on $\partial\mathbb{D}$.

(b) Let log be the principal branch of the logarithm, and γ the positively oriented arc $\{e^{it} : t \in [0, \pi/2]\}$. What is the value of

$$\int_\gamma \log(z^2)\, dz?$$

A) $2i$

B) $\pi - 2 + 2i$

C) $\pi + 2 - i$

D) $2 - 2i - \pi$

12.2. Holomorphic injections Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic injection.

(a) Let $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be defined as $g(z) := f(1/z)$. Show that g has no essential singularity at zero. Argue by contradiction taking advantage of the Theorem of Casorati-Weierstrass [1].

(b) Taking advantage of the Laurent series of f and g, prove that f is in fact a polynomial.

(c) Show that f is of the form $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

12.3. Estimates on the modulus Suppose that f is holomorphic on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$, $f(i) = 0$, and that $|f(z)| \leq 1$ for all $z \in \mathbb{H}$. How big can $|f(2i)|$ be under these conditions?

[1] Recall: If $f : B(a, R) \setminus \{a\} \to \mathbb{C}$ holomorphic has an essential singularity in a, then for all $0 < r < R$, $f(B(a, r) \setminus \{a\})$ is dense in $\mathbb{C}$. (Here $B(w, \rho) := \{z \in \mathbb{C} : |z - w| < \rho\}$).

12.4. Schwarz-Pick's Lemma Denote with $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in $\mathbb{C}$, and suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Prove that for all $z \in \mathbb{D}$

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Remark: note that the above expression takes the nicer form: $|f'(a)| \leq \frac{1-|b|^2}{1-|a|^2}$, *for all* $a, b \in \mathbb{D}$ *such that* $f(a) = b$.

12.5. Symmetries of the Riemann mapping Let $\Omega \subset \mathbb{C}$ be a non-empty, simply connected domain symmetric with respect to the real axis ($\{\bar{z} : z \in \Omega\} = \Omega$). For $z_0 \in \Omega$ real, denote with $F : \Omega \to \mathbb{D}$ the unique conformal map given by the Riemann Mapping Theorem, so that $F(z_0) = 0$ and $F'(z_0) > 0$. Prove that

$$\overline{F(\bar{z})} = F(z),$$

for all $z \in \Omega$.

Hint: take advantage of Exercise 7.2. on the Schwarz reflection principle.