1. Multiple Choice Questions Exactly one answer to each MC question is correct.

(a) Which of the following is **not** enough to conclude that f is constant?

- ◯ $\Omega = \mathbb{D}$ and $f(x) = \pi$ for all $x \in (-1, 1)$.
- ◯ $\Omega = \mathbb{C}$ and $f(\mathbb{C}) \subset \mathbb{D}$.
- ◯ $\Omega = \mathbb{C}$ and $|f(z)| < \log(1 + |z|)$ for $|z| > 2024$.
- ◯ $\Omega = \mathbb{C} \setminus \{0\}$ and $f(1/n) = 0$ for all $n \in \mathbb{N}$.

(b) Which of the following statements is **not** correct?

- ◯ If f has an essential singularity at $z_0 = 0$, then for every $w \in \mathbb{C}$, there exists a sequence (z_n) in the image of f such that $f(z_n) \to 0$ and $z_n \to w$.
- ◯ $\int_{|z|=1} \frac{1}{z(\cos(z))^2}\, dz = 2\pi i$.
- ◯ $f(z) = \frac{(\sin(z))^3}{z^3(z+5)}$ has simple poles at $z = 0$ and $z = -5$.
- ◯ If f and g both have a zero at z_0 of order 5, then the function fg has a zero at z_0 of order 10.

(c) Which $\Omega \subset \mathbb{C}$ is **not** biholomorphic to $\mathbb{C} \setminus [0, +\infty)$?

- ◯ $\Omega = \{z \in \mathbb{C} : \Im(z) > \Re(z)^2\}$.
- ◯ $\Omega = \mathbb{C}$.
- ◯ $\Omega = \{z \in \mathbb{C} : \Re(z) \in (-1, 1)\}$.
- ◯ $\Omega = \{z \in \mathbb{C} : |z + i| < 2\}$.

(d) Let $f(z) = z^6 + 7z^3 - 2z^2 + 3$. How many zeroes does f have inside the open unit disk $\mathbb{D}$?

- ◯ f has exactly 3 zeroes in $\mathbb{D}$.
- ◯ f has exactly 4 zeroes in $\mathbb{D}$.
- ◯ f has exactly 5 zeroes in $\mathbb{D}$.
- ◯ f has exactly 6 zeroes in $\mathbb{D}$.

(e) In which open set does the following map define a holomorphic function?

$$f(z) = \sum_{n \geq 1} \Big(2i \cos(\pi n)\Big)^{4n} z^{-2n}.$$

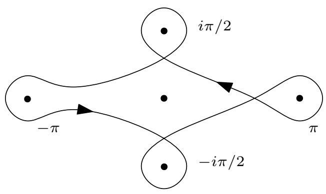
- ◯ $\{z \in \mathbb{C} : 4 < |z|\}$.

- ◯ $\{z \in \mathbb{C} : 2 < |z|\}$.
- ◯ $\{z \in \mathbb{C} : 0 < |z| < 2\}$.
- ◯ $\{z \in \mathbb{C} : 0 < |z| < \sqrt{2}\}$.

(f) Which formula does **not** hold for every non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$?

- ◯ $\int_{|z|=1} \sin(f(z)^2)\, dz = 0$.
- ◯ $\frac{1}{2\pi i} \int_{|z|=1} \frac{1}{f(z)z^2}\, dz = -\frac{f'(0)}{f(0)^2}$.
- ◯ $\frac{1}{2\pi i} \int_{|z|=1} \frac{1}{f(z)-f(0)}\, dz = \operatorname{res}_{z=0} \frac{1}{f}$.
- ◯ $\int_{|z|=1} \frac{f'(z)}{f(z)}\, dz = 0$.

2. Open Question Consider the meromorphic function $f(z) = \frac{z^2}{\sin(z)\cos(iz)}$, and the curve γ as in the following figure:



(a) Find **all** zeroes and poles of f and their order.

(b) Compute the integral $\int_\gamma f\, dz$.

3. Open Question Show that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\cos(\theta)}\, d\theta = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{(z+z^{-1})/2}}{z}\, dz = \sum_{n=0}^{+\infty} \left(\frac{1}{2^n n!}\right)^2.$$

Hint: take advantage of the series representation of the exponential: $e^w = \sum_{n=0}^{+\infty} \frac{w^n}{n!}$.

4. Open Question Let f be a holomorphic map of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto itself and such that $f(0) = 0$. Show that

$$|f(z) + f(-z)| \leq 2|z|^2,$$

for all $z \in \mathbb{D}$.

Recall Schwarz Lemma: Let $f : \mathbb{D} \to \mathbb{D}$ *analytic,* $f(0) = 0$. *Then,* $|f'(0)| \leq 1$, $|f(z)| \leq |z|$, $\forall z \in \mathbb{D}$.

5. Open Question Let $f : \mathbb{D} \to \hat{\mathbb{C}}$ be **any** function, and suppose that $g = f^2$ and $h = f^3$ are meromorphic functions. Let Z_f, Z_g, Z_h and P_f, P_g, P_h the set of zeros and poles of f, g, h respectively without taking into account their multiplicity.

(a) Show that f is meromorphic in $\mathbb{D}$.

(b) Determine all poles and zeros of f and their orders in terms of the poles and zeros of h and g. Show that $Z_f = Z_g = Z_h$ and $P_f = P_g = P_h$. What can you say about the orders of the poles and zeros?

(c) Show that if h is holomorphic in $\mathbb{D}$ the same holds for f.